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THE ELEMENTS OF EUCLID.

THE

ELEMENTS OF EUCLID,

(The parts read in the University of Cambridge)

FROM THE TEXT OF

DR. ROBERT SIMSON,

WITH GEOMETRICAL PROBLEMS FOR SOLUTION,

BY

THE REV. J. W. COLENSO, M.A.

LATE FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE:
RECTOR OF FORNCETT ST. MARY, NORFOLK.

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DEIGHTONS, CAMBRIDGE.

1846.

867.

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ADVERTISEMENT.

This edition of Euclid has been printed, with some slight verbal emendations, from the text of Dr. Simson, except that, in 5. 18, I have substituted a known shorter demonstration for the long and cumbrous direct proof. As this Proposition is one of those often omitted, there seemed less reason for retaining this latter. I have throughout attended carefully to the punctuation, and, I hope, have considerably improved it. It will be found that the different steps of the reasoning are distinctly separated by means of a colon and a capital. For the rest of the Text of Euclid and additional Problems, reference may be made to Potts' Edition of Euclid, where also is given much additional matter, of use to the Tutor and Private Student.

Forncett St. Mary, Oct. 31, 1846.

ERRATA.

Page	
48	PROB. 3, for *bisect* read *be bisected by*.
39	Prob. 10, *Take* D *any point between* B *and the point where a circle, centre* C, *and rad*. CA, *the* least *of the two sides, cuts* AB.
51	Prob. 36, after *point* insert *between them*.
52	Prob. 51, for *the sides* read *the alternate sides*.
53	Prob. 58, for *bisect* read *trisect*.
53	Prob. 60, for *greater or less* read *less or greater*.

N.B. *In the* PROBLEMS, *the name trapezium will be restricted to a quadrilateral, two of whose opposite sides are parallel.*

THE

ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS.

I. A POINT is that which hath no parts, or which hath no magnitude.

II. A line is length without breadth.

III. The extremities of a line are points.

IV. A straight line is that which lies evenly between its extreme points.

V. A superficies is that which hath only length and breadth.

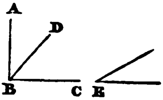
VI. The extremities of a superficies are lines.

VII. A plane superficies is that in which, any two points being taken, the straight line between them lies wholly in that superficies.

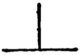
VIII. A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

IX. A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

N.B. When several angles are at one point B, any one of them is expressed by three letters, of which the letter that is at the vertex of the angle, that is, at the point in which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two is somewhere upon one of those straight lines, and the other upon the other line; thus the angle which is contained by the straight lines AB, CB, is named the angle ABC or CBA, that which is contained by AB, DB, is named the angle ABD or DBA, and that which is contained by DB, CB, is named the angle DBC or CBD: but, if there be only one angle at a point, it may be expressed by a letter placed at that point, as the angle at E.



X. When a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of these angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.



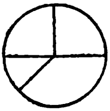
XI. An obtuse angle is that which is greater than a right angle.

XII. An acute angle is that which is less than a right angle.

XIII. A term or boundary is the extremity of any thing.

XIV. A figure is that which is inclosed by one or more boundaries.

XV. A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines, drawn from a certain point within the figure to the circumference, are equal to one another:



XVI. And this point is called the centre of the circle.

XVII. A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

XVIII. A semicircle is the figure contained by a diameter and the part of the circumference it cut off.

XIX. A segment of a circle is the figure contained by a straight line, and the circumference it cuts off.

XX. Rectilineal figures are those which are contained by straight lines:

XXI. Trilateral figures, or triangles, by three straight lines:

XXII. Quadrilateral, by four straight lines:

XXIII. Multilateral figures, or polygons, by more than four straight lines:

Of three-sided figures,

XXIV. An equilateral triangle is that which has three equal sides:

XXV. An isosceles triangle is that which has two sides equal:

XXVI. A scalene triangle is that which has three unequal sides:

XXVII. A right-angled triangle is that which has a right angle:

XXVIII. An obtuse-angled triangle is that which has an obtuse angle:

XXIX. An acute-angled triangle is that which has three acute angles.

Of four-sided figures,

xxx. A square is that which has all its sides e and all its angles right angles :

xxxi. An oblong is that which has all its a right angles, but not all its sides equal :

xxxii. A rhombus is that which has all its si equal, but its angles are not right angles :

xxxiii. A rhomboid is that which has its oppos sides equal to one another, but all its sides are n equal, nor its angles right angles.

xxxiv. All other four-sided figures besides these a called Trapeziums.

xxxv. Parallel straight lines are such as are in th same plane, and, being produced ever so far both ways, do not meet.

POSTULATES.

Let it be granted,

i. That a straight line may be drawn from any one point to any other point :

ii. That a terminated straight line may be produced to any length in a straight line :

iii. That a circle may be described from any centre, at any distance from that centre.

AXIOMS.

i. Things which are equal to the same are equal to one another.

II. If equals be added to equals, the wholes are equal.

III. If equals be taken from equals, the remainders are equal.

IV. If equals be added to unequals, the wholes are unequal.

V. If equals be taken from unequals, the remainders are unequal.

VI. Things which are double of the same, are equal to one another.

VII. Things which are halves of the same, are equal to one another.

VIII. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

IX. The whole is greater than its part.

X. Two straight lines cannot enclose a space.

XI. All right angles are equal to one another.

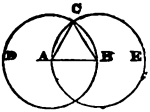
XII. If a straight line falls upon two straight lines, so as to make the two interior angles on the same side of it together less than two right angles, these straight lines, being continually produced, shall at length meet, upon that side on which are the angles which are less than two right angles.

PROPOSITION I. PROBLEM.

To describe an equilateral triangle upon a given finite straight line.

Let AB be the given straight line: it is required to describe an equilateral triangle upon AB.

From the centre A, at the distance AB, describe the circle BCD (*Post.* 3), and from the centre B, at the distance BA, describe the circle ACE; and from the point C, in which the circles cut one another, draw the straight lines CA, CB, to the points A, B (*Post.* 1): ABC shall be an equilateral triangle.



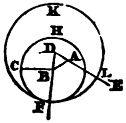
Because the point A is the centre of the circle BCD, AC is equal to AB (*Def.* 15); and because the point B is the centre of the circle ACE, BC is equal to BA: But it has been proved that CA is equal to AB; therefore CA, CB are each of them equal to AB: But things which are equal to the same are equal to one another (*Ax.* 1); therefore CA is equal to CB: Wherefore CA, AB, BC are equal to one another; and the triangle ABC is therefore equilateral, and it is described upon the given straight line AB. Q. E. F.

PROP. II. PROB.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line: it is required to draw from the point A a straight line equal to BC.

From the point A to B draw the straight line AB; and upon it describe the equilateral triangle DAB (I. 1), and produce the straight lines DA, DB, to E and F (*Post.* 2); from the centre B, at the distance BC, describe the circle CGH, and from the centre D, at the distance DG, describe the circle GKL: AL shall be equal to BC.

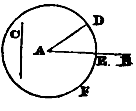


Because the point B is the centre of the circle CGH, BC is equal to BG: And because D is the centre of the circle GKL, DL is equal to DG; and DA, DB, parts of them, are equal (*Constr.*), therefore the remainder AL is equal to the remainder BG (*Ax.* 3): But it has been shewn that BC is equal to BG; therefore AL and BC are each of them equal to BG: And things which are equal to the same are equal to one another; therefore the straight line AL is equal to BC. Wherefore from the given point A a straight line AL has been drawn equal to the given straight line BC. Q. E. F.

PROP. III. PROB.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, whereof AB is the greater: it is required to cut off from AB the greater a part equal to C the less.



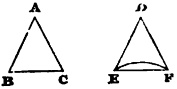
From the point A draw the straight line AD equal to C (I. 2); and from the centre A, at the distance AD, describe the circle DEF.

Then, because A is the centre of the circle DEF, AE is equal to AD: But the straight line C is likewise equal to AD (*Constr.*); whence AE and C are each of them equal to AD: Wherefore AE is equal to C, and from AB, the greater of two given straight lines, a part AE has been cut off equal to C the less. Q.E.F.

PROP. IV. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases, or third sides, *equal, and the two triangles shall be equal, and their other angles shall be equal, each to each, viz. those to which the equal sides are opposite.*

Let ABC, DEF be two triangles, which have the two sides AB, AC equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF, and the angle BAC equal to the angle EDF: the base BC shall be equal to the base EF, and the triangle ABC to the triangle DEF, and the other angles shall be equal, each to each, to which the equal sides are opposite, viz. the angle ABC to the angle DEF, and the angle ACB to the angle DFE.



For, if the triangle ABC be applied to the triangle DEF, so that the point A may be on D, and the straight line AB upon DE, the point B shall coincide with the point E, because AB is equal to DE (*Hyp.*): and AB coinciding with DE, AC shall coincide with DF, because the angle BAC is equal to the angle EDF (*Hyp.*): therefore also the point C shall coincide with the point F, because AC is equal to DF (*Hyp.*): But the point B coincides with the point E, therefore the base BC shall coincide with the base EF; because, B coinciding with E, and C with F, if BC does not coincide with EF, two straight lines would enclose a space—which is impossible (*Ax.* 10); therefore the base BC shall coincide with the base EF, and be equal to it (*Ax.* 8); and the whole triangle ABC shall coincide with the whole triangle DEF, and be equal to it; and the other angles of the one shall coincide with the other angles of the other, and be equal to them, each to each, viz. the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

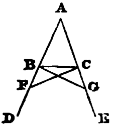
Wherefore, *If two triangles &c.* Q. E. D.

PROP. V. THEOR.

The angles at the base of an isosceles triangle are equal to one another; and, if the equal sides be produced, the angles upon the other side of the base shall be equal.

Let ABC be an isosceles triangle, of which the side AB is equal to AC, and let the straight lines AB, AC be produced to D and E: the angle ABC shall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F, and from AE the greater cut off AG equal to AF the less (I. 3), and join FC, GB.



Then, because AF is equal to AG, and AB to AC (*Hyp.*), the two sides FA, AC are equal to the two GA, AB, each to each, and they contain the angle FAG common to the two triangles AFC, AGB—therefore (I. 4) the base FC is equal to the base GB, and the triangle AFC to the triangle AGB, and the remaining angles of the one to the remaining angles of the other, each to each, to which the equal sides are opposite, viz. the angle ACF to the angle ABG and the angle AFC to the angle AGB:

And because the whole AF is equal to the whole AG, of which the parts AB, AC are equal, the remainder BF is equal to the remainder CG, and FC was proved to be equal to GB; therefore the two sides BF, FC are equal to the two CG, GB, each to each, and the angle BFC has been proved to be equal to the angle CGB—therefore (I. 4) (the base BC being here common to the two triangles BFC, CGB) the two triangles are equal, and their other angles are equal, each to each, to which the equal sides are opposite, viz. the angle FBC to the angle GCB and the angle BCF to the angle CBG:

And, since it has been demonstrated that the whole angle ABG is equal to the whole angle ACF, parts of which, the angles CBG, BCF, are also equal, the remaining angle ABC is therefore equal to the remaining angle ACB (*Ax.* 3), which are the angles at the base of the triangle ABC: And it has been proved that the angle FBC is equal to the angle GCB, which are the angles upon the other side of the base.

Wherefore, *The angles at the base &c.* Q. E. D.

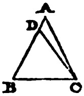
COROLLARY. Hence every equilateral triangle is also equiangular.

PROP. VI. THEOR.

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, *the equal angles, shall be equal to one another.*

Let ABC be a triangle having the angle ABC equal to the angle ACB: the side AB shall be equal to the side AC.

For, if AB be not equal to AC, one of them is greater than the other. Let AB be the greater; and from it cut off DB equal to AC the less, and join DC: Then, because in the triangles DBC, ACB, DB is equal to AC, and BC common to both triangles, the two sides DB, BC are equal to the two AC, CB, each to each, and the angle DBC is equal to the angle ACB (*Hyp.*)—therefore the base DC is equal to the base AB (I. 4), and the triangle DBC is equal to the triangle ACB, the less to the greater—which is absurd: Therefore AB is not unequal to AC, that is, it is equal to it.



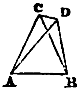
Wherefore, *If two angles &c.* Q. E. D.

COR. Hence every equiangular triangle is also equilateral.

PROP. VII. THEOR.

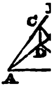
Upon the same base, and on the same side of it, there cannot be two triangles which have their sides terminated in one extremity of the base equal to one another, and also those terminated in the other extremity.

If it be possible, let there be two triangles ACB, ADB, upon the same base AB, and upon the same side of it, which have their sides CA, DA terminated in the extremity A of the base equal to one another, and also their sides CD, DB, equal, terminated in B.



Join CD: Then, in the case in which the vertex of each of the triangles is *without* the other triangle, because AC is equal to AD, the angle ACD is equal to the angle ADC (I. 5): But the angle ACD is greater than the angle BCD (*Ax.* 9); therefore also the angle ADC is greater than the angle BCD; much more then is the angle BDC greater than the a BCD: Again, because BC is equal to BD the a BCD is equal to the angle BDC: but it has proved to be greater than it—which is impossible.

But, if one of the vertices, as D, be *within* the c triangle ACB, produce AC, AD to E, F. Then, because AC is equal to AD in the triangle ACD, the angles ECD, FDC upon the other side of the base CD are equal to one another (I. 5): But the angle ECD is greater than the angle BCD; therefore also the FDC is greater than the angle BCD; much then is the angle BDC greater than the angle B Again, because BC is equal to BD, the angle BC equal to the angle BDC: but it has been proved greater than it—which is impossible.



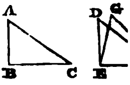
The case, in which the vertex of one triangle is *a side* of the other, needs no demonstration.

Wherefore, *Upon the same base, &c.* Q.E.D.

PROP. VIII. THEOR.

If two triangles have two sides of the one equal to two of the other, each to each, and have also their equal, the angle which is contained by the two si the one shall be equal to the angle contained by th sides, equal to them, of the other.

Let ABC, DEF be two triangles, having the two AB, AC equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF, and also the base BC equal to the base EF: the angle BAC shall be equal to the angle EDF.



For, if the triangle ABC be applied to the triangle DEF, so that the point B may be on E, and the straight line BC upon EF, the point C shall also coincide with the point F, because BC is equal to EF (*Hyp.*) : And BC coinciding with EF, BA and AC shall coincide with ED and DF; for, if the base BC coincides with the base EF, whereas the sides BA, CA do not coincide with the sides ED, FD, but have a different situation as EG, FG, then upon the same base EF, and upon the same side of it, there would be two triangles having their sides terminated in one extremity of the base equal to one another, and also those terminated in the other extremity—which is impossible (I. 7) : Therefore, if the base BC coincides with the base EF, the sides BA, AC cannot but coincide with the sides ED, DF; and therefore also the angle BAC coincides with the angle EDF, and is equal to it.

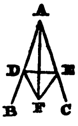
Wherefore, *If two triangles &c.* Q.E.D.

PROP. IX. PROB.

To bisect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle: it is required to bisect it.

Take any point D in AB, and from AC cut off AE equal to AD; join DE, and upon the side opposite to that on which A lies, describe an equilateral triangle DEF, and join AF: the straight line AF bisects the angle BAC.



Because AD is equal to AE, and AF is common to the two triangles DAF, EAF, the two sides DA, AF are equal to the two sides EA, AF, each to each, and the base DF is equal to the base EF

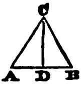
--therefore (I. 8) the angle DAF is equal to the angle EAF: Wherefore the given rectilineal angle BAC is bisected by the straight line AF. Q. E. F.

PROP. X. Prob.

To bisect a given finite straight line, that is, to divide it into two equal parts.

Let AB be the given straight line: it is required to bisect it.

Upon it describe an equilateral triangle ABC, and bisect the angle ACB by the straight line CD (I. 9): the straight line AB is bisected in the point D.

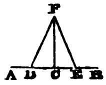


Because AC is equal to CB and CD is common to the two triangles ACD, BCD, the two sides AC, CD are equal to the two BC, CD, each to each, and the angle ACD is equal to the angle BCD (*Constr.*)—therefore the base AD is equal to the base DB: Wherefore the straight line AB is bisected in the point D. Q. E. F.

PROP. XI. Prob.

To draw a straight line at right angles to a given straight line from a given point in the same.

Let AB be the given straight line, and C a given point in it: it is required to draw from the point C a straight line at right angles to AB.

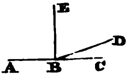


Take any point D in AC, and make CE equal to CD (I. 3), and upon DE describe the equilateral triangle DFE, and join CF: the straight line CF is drawn from the given point C at right angles to the given straight line AB.

Because DC is equal CE, and CF common to the two triangles DCF, ECF, the two sides DC, CF are equal to the two EC, CF, each to each, and the base DF is equal to the base EF—therefore (I. 8) the angle DCF is equal to the angle ECF; and they are adjacent angles: But, *when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of these angles is called a right angle* (*Def.* 10); therefore each of the angles DCF, ECF is a right angle: Wherefore, from the given point C, in the given straight line AB, CF has been drawn at right angles to AB. Q.E.F.

COR. By the help of this problem, it may be demonstrated that two straight lines cannot have a common segment.

If it be possible, let the two straight lines ABC, ABD have the segment AB common to both of them. From the point B draw BE at right angles to AB: Then, because ABC is a straight line, the angle CBE is equal to the angle EBA (*Def.* 10): So also, because ABD is a straight line, the angle DBE is equal to the angle EBA: Therefore the angle DBE is equal to the angle CBE, the less to the greater—which is absurd: Wherefore two straight lines cannot have a common segment.

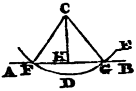


PROP. XII. PROB.

To draw a straight line, perpendicular to a given straight line of unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced to any length both ways, and let C be a point without it: it is required to draw from the point C a straight line perpendicular to AB.

Take any point D upon the other side of AB, and from the centre C, at the distance CD, describe the circle EGF meeting AB in F, G; bisect FG in H (I. 10), and join CF, CH, CG: the straight line CH is drawn from the given point C perpendicular to the given straight line AB.



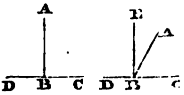
Because FH is equal to HG, and HC common to the two triangles FHC, GHC, the two sides FH, HC are equal to the two GH, HC, each to each, and the base CF is equal to the base CG—therefore (I. 8) the angle CHF is equal to the angle CHG; and they are adjacent angles: But, *when a straight line standing on another straight line makes the adjacent angles equal to one another, each of these angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it* (*Def.* 10); therefore CH is perpendicular to AB: Wherefore, from the given point C without it, CH has been drawn perpendicular to the given straight line AB. Q.E.F.

PROP. XIII. THEOR.

The angles, which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line AB make with CD, upon one side of it, the angles CBA, ABD: these are either two right angles, or are together equal to two right angles.

For if the angle CBA be equal to ABD, each of them is a right angle (*Def.* 10): But if not, from the point B draw BE at right angles to CD (I. 11); then the angles CBE, EBD are two right angles: And, because

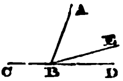


the angle CBE is equal to the two angles CBA, ABE, to each of these equals add the angle EBD; therefore the angles CBE, EBD are equal to the three angles CBA, ABE, EBD (*Ax.* 2): Again, because the angle DBA is equal to the two angles DBE, EBA, to each of these equals add the angle ABC; therefore the angles DBA, ABC are equal to the three angles DBE, EBA, ABC: But the angles CBE, EBD have been shewn to be equal to the same three angles; and *things that are equal to the same are equal to one another;* therefore the angles CBE, EBD are equal to the angles DBA, ABC: But CBE, EBD are two right angles; therefore DBA, ABC are together equal to two right angles.

Wherefore, *The angles, which one straight line &c.* Q. E. D.

PROP. XIV. THEOR.

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.



At the point B in the straight line AB, let the two straight lines BC, BD, upon the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles: BD shall be in the same straight line with BC.

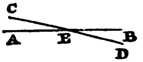
For, if BD be not in the same straight line with BC, let BE be in the same straight line with it: Then because the straight line AB makes with the straight line CBE, upon one side of it, the angles CBA, ABE, these are together equal to two right angles (I. 13): But the angles CBA, ABD are together equal to two right angles (*Hyp.*); therefore the angles CBA, are equal to the angles CBA, ABD: From e these equals take away the common angle CBA; the remaining angle ABE is equal to the rema angle ABD (*Ax.* 3), the less to the greater—whi absurd: Therefore BE is not in the same straight with BC: And in like manner it may be den strated, that no other can be in the same straight with it but BD: Therefore BD is in the same strai line with BC.

Wherefore, *If at a point &c.* Q.E.D.

PROP. XV. THEOR.

If two straight lines cut one another, the vertical, or opposite, *angles shall be equal.*

Let the two straight lines AB, CD cut one anothe in the point E: the angle AEC shall be equal to the angle BED, and the angle AED to the angle BEC.



Because the straight line AE makes with CD the angles AEC, AED, these are together equal to two right angles (I. 13): Again, because the straight line DE makes with AB the angles AED, BED, these also are together equal to two right angles: But the angles AEC, AED have been shewn to be equal to two right angles; therefore the angles AEC, AED, are equal to the angles AED, BED: From each of these equals take away the common angle AED; then the remaining angle AEC is equal to the remaining angle BED.

And in like manner it may be demonstrated that the angle AED is equal to the angle BEC.

Wherefore, *If two straight lines &c.* Q.E.D.

Cor. 1. From this it is manifest that, if two straight lines cut one another, the angles they make at the point where they cut are together equal to four right angles:

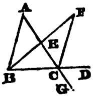
Cor. 2. And, consequently, that all the angles, made by any number of lines meeting in one point, are together equal to four right angles.

PROP. XVI. Theor.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Let ABC be a triangle, and let its side BC be produced to D: the exterior angle ACD is greater than either of the interior opposite angles CBA, BAC.

Bisect AC in E (I. 10), join BE and produce it to F, making EF equal to BE (I. 3); and join FC.



Then, because AE is equal to EC, and BE to EF, the two sides, AE, EB, are equal to the two, CE, EF, each to each, and the angle AEB is equal to the angle CEF, because they are opposite vertical angles (I. 15);—therefore the base AB is equal to the base CF, and the triangle AEB to the triangle CEF, and the remaining angles to the remaining angles, each to each, to which the equal sides are opposite; therefore the angle BAE is equal to the angle ECF: But the angle ECD is greater than the angle ECF; therefore the angle ACD is greater than the angle BAE, or BAC.

And, in like manner, if the side BC be bisected and AC be produced to G, it may be demonstrated that the angle BCG, that is, the angle ACD (I. 15.), is greater than the angle ABC.

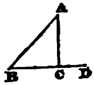
Wherefore, *If one side &c.* Q.E.D.

PROP. XVII. THEOR.

Any two angles of a triangle are together less than two right angles.

Let ABC be a triangle: any two of its angles are together less than two right angles.

Produce BC to D: Then, because ACD is the exterior angle of the triangle ABC, ACD is greater than the interior and opposite angle ABC (I. 16): To each of these add the angle ACB; therefore the angles ACD, ACB, are together greater than the angles ABC, ACB (*Ax.* 4): But the angles ACD, ACB, are together equal to two right angles (I. 13); therefore the angles ABC, ACB are together less than two right angles. And, in like manner, it may be demonstrated that the angles BAC, ACB, as also the angles CAB, ABC, are together less than two right angles.



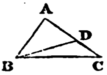
Wherefore, *Any two angles &c.* Q. E. D.

PROP. XVIII. THEOR.

The greater side of every triangle is opposite to the greater angle.

Let ABC be a triangle, of which the side AC is greater than the side AB: the angle ABC is also greater than the angle ACB.

Since AC is greater than AB, cut off AD equal to AB, and join BD: Then, because ADB is the exterior angle of the triangle BDC, it is greater than the interior and opposite angle DCB (I. 16): But the angle ADB is equal to the angle ABD, because the side AD is equal to the side AB (I. 5); therefore also the angle



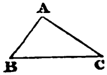
ABD is greater than the angle ACB: much more then is the angle ABC greater than the angle ACB.

Wherefore, *The greater side &c.* Q. E. D.

PROP. XIX. THEOR.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Let ABC be a triangle, of which the angle ABC is greater than the angle ACB: the side AC is also greater than the side AB.



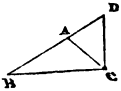
For, if it be not greater, it must either be equal to it, or less: But AC is not equal to AB, because then the angle ACB would be equal to the angle ABC (I. 5); but it is not—therefore AC is not equal to AB: Neither is it less, because then the angle ACB would be less than the angle ABC (I. 18); but it is not—therefore AC is not less than AB: And it has been shewn that it is not equal to it; therefore AC is greater than AB.

Wherefore, *The greater angle &c.* Q. E. D.

PROP. XX. THEOR.

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle: any two sides of it are together greater than the third side, viz. BA, AC greater than BC, and AB, BC, greater than AC, and AC, CB, greater than AB.



Produce BA to the point D, making AD equal to AC (I. 3), and join DC.

Then because AD is equal to AC, the angle ADC is equal to the angle ACD:

But the angle BCD is greater than the angle ACD; therefore the angle BCD is greater than the angle BDC: And because the angle BCD of the triangle BCD is greater than the angle BDC of the same triangle, and that the greater side is opposite to the greater angle (I. 19), therefore the side BD is greater than the side BC: But BD is equal to BA and AC; therefore BA, AC are greater than BC.

And in like manner it may be demonstrated that AB, BC, are greater than AC, and AC, CB, greater than AB.

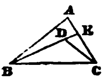
Wherefore, *Any two sides &c.* Q. E. D.

PROP. XXI. THEOR.

If from the ends of a side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let the two straight lines BD, CD be drawn from B, C, the ends of the side BC of the triangle ABC, to the point D within it: BD and DC are less than the other two sides BA, AC of the triangle, but contain an angle BDC greater than the angle BAC.

Produce BD to E: Then, because any two sides of a triangle are together greater than the third side (I. 20), the two sides, BA, AE, of the triangle ABE are greater than BE: To each of these add EC; therefore the sides BA, AC are greater than BE, EC: Again, because the two sides, CE, ED, of the triangle CED are greater than CD, to each of these add DB; therefore the sides CE, EB are greater than CD, DB: But it has been shewn that BA, AC are greater than BE, EC; much more then are BA, AC greater than BD, DC.



Again, because the exterior angle of a triangle is greater than the interior and opposite angle (I. 16), the exterior angle BDC of the triangle CDE is greater than the angle CED: For the same reason, the exterior angle CEB of the triangle ABE is greater than the angle BAE: But it has been shewn that the angle BDC is greater than the angle CEB; much more then is the angle BDC greater than the angle BAC.

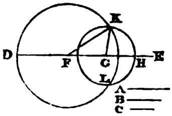
Wherefore, *If from the ends &c.* Q. E. D.

PROP. XXII. PROB.

To make a triangle, of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third (I. 20).

Let A, B, C, be the three given straight lines, of which any two whatever are greater than the third, viz. A and B greater than C, A and C greater than B, and B and C greater than A: it is required to make a triangle, of which the sides shall be equal to A, B, C, each to each.

Take a straight line DE terminated at the point D, but unlimited towards E, and (I. 3) make DF equal to A, FG equal to B, and GH equal to C; from the centre F, at the distance FD, describe the circle DKL, and from the centre G, at the distance GH, describe the circle HLK; and join KF, KG: the triangle KFG has its sides equal to the three straight lines A, B, C.



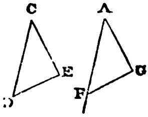
Because the point F is the centre of the circle DKL, FD is equal to FK: but FD was made equal to A; therefore also FK is equal to A: Again, because G is the centre of the circle HLK, GH is equal to GK: but GH was made equal to C; therefore also GK is equal to C: And FG is equal to B; therefore the three straight lines KF, FG, GK, are equal to the three A, B, C; and therefore the triangle KFG has its three sides KF, FG, GK, equal to the three given straight lines A, B, C. Q.E.F.

PROP. XXIII. PROB.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle: it is required to make at the given point A in the given straight line AB an angle, equal to the given rectilineal angle DCE.

In CD, CE take any points D, E, and join DE; and make the triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, EC (I. 22), so that AF shall be equal to CD, AG to CE, and FG to DE: the angle FAG shall be equal to the angle DCE.



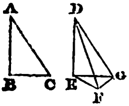
For because DC, CE, are equal to FA, AG, each to each, and the base DE to the base FG—therefore (I. 8) the angle DCE is equal to the angle FAG: Wherefore, at the given point A in the given straight line AB, the angle FAG has been made equal to the given rectilineal angle DCE. Q.E.F.

PROP. XXIV. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of the one greater than the angle contained by the two sides, equal to them, of the other, the base of that which has the greater angle shall be greater than the base of the other.

Let ABC, DEF be two triangles, which have the two sides AB, AC equal to the two DE, DF, each to each, viz. AB to DE, and AC to DF, but the angle BAC greater than the angle EDF: the base BC shall be also greater than the base EF.

Of the two sides DE, DF, let DE be the side which is not greater than the other, and at the point D, in the straight line DE, make the angle EDG equal to the angle BAC (1. 23); make DG equal to AC or DF, and join EG, GF.



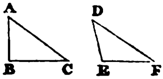
Because AB is equal to DE, and AC to DG, the two sides BA, AC are equal to the two ED, DG, each to each, and the angle BAC is equal to the angle EDG—therefore the base BC is equal to the base EG And because DF is equal to DG, the angle DFG is equal to the angle DGF (1. 5): But the angle DGF is greater than the angle EGF; therefore the angle DFG is greater than EGF; much more then is the angle EFG greater than the angle EGF: And because the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater side is opposite to the greater angle (1. 19)—therefore the side EG is greater than the side EF: But BC was proved to be equal to EG; therefore also BC is greater than EF.

Wherefore, *If two triangles &c.* Q.E.D.

PROP. XXV. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base shall be greater than the angle contained by the sides, equal to them, of the other.

Let ABC, DEF be two triangles, which have the two sides AB, AC equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF, but the base BC greater than the base EF: the angle BAC shall be greater than the angle EDF.



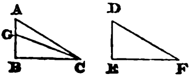
For, if it be not greater, it must be either equal to it, or less: But the angle BAC is not equal to the angle EDF, because then the base BC would be equal to the base EF (I. 4); but it is not—therefore the angle BAC is not equal to the angle EDF: Neither is it less, because then the base BC would be less than the base EF (I. 24); but it is not—therefore the angle BAC is not less than the angle EDF: And it has been shewn that it is not equal to it; therefore the angle BAC is greater than the angle EDF.

Wherefore, *If two triangles &c.* Q.E.D.

PROP. XXVI. THEOR.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, viz. either the sides adjacent to the equal angles, or sides which are opposite to equal angles in each, then shall the other sides be equal, each to each, and also the third angle of the one to the third angle of the other.

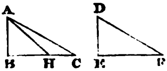
Let ABC, DEF be two triangles, which have the angles ABC, ACB, equal to the angles DEF, DFE, each to each, viz. ABC to DEF, and ACB to DFE, and have also one side equal to one side, and first, let those sides be equal which are adjacent to the angles that are equal in the two triangles, viz. BC to EF: then the other sides shall be



equal, each to each, viz. AB to DE, and AC to DF, and the third angle BAC to the third angle EDF.

For, if AB be not equal to DE, one of them must be greater than the other. Let AB be the greater, and make BG equal to DE, and join GC: Then, because BG is equal to DE, and BC to EF (*Hyp.*), the two sides GB, BC are equal to the two DE, EF, each to each, and the angle GBC is equal to the angle DEF—therefore the base GC is equal to the base DF, and the triangle GBC to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle GCB is equal to the angle DFE: But the angle DFE is equal to the angle ACB (*Hyp.*); therefore also the angle GCB is equal to the angle ACB, the less to the greater—which is absurd: Therefore AB is not unequal to DE, that is, it is equal to it; and BC is equal to EF; therefore the two AB, BC are equal to the two DE, EF, each to each, and the angle ABC is equal to the angle DEF—therefore the base AC is equal to the base DF, and the third angle BAC to the third angle EDF.

Next, let sides which are opposite to equal angles in each triangle be equal to one another, viz. AB to DE: likewise in this case, the other sides shall be equal, viz. BC to EF, and AC to DF, and also the third angle BAC to the third angle EDF.



For if BC be not equal to EF, let BC be the greater, and make BH equal to EF, and join AH: Then because BH is equal to EF, and AB to DE (*Hyp.*), the two AB, BH are equal to the two DE, EF, each to each, and they contain equal angles (*Hyp.*)—therefore the base AH is equal to the base DF, and the triangle ABH to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle AHB is equal to the angle DFE: But the angle DFE is equal to the angle ACB (*Hyp.*); therefore also the angle AHB is equal to the angle ACB, that is, the exterior angle AHB of the triangle AHC is equal to its interior and opposite angle ACB—which is impossible (I. 16): Therefore BC is not unequal to EF, that is, it is equal to it; and AB is equal to DE (*Hyp.*); therefore the two, AB, BC, are equal to the two, DE, EF, each to each, and they contain equal angles—therefore the base AC is equal to the base DF, and the third angle BAC to the third angle EDF.

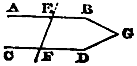
Wherefore, *If two triangles &c.* Q.E.D.

PROP. XXVII. THEOR.

If a straight line, falling upon two other straight lines, makes the alternate angles equal to one another, these two straight lines shall be parallel.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the alternate angles AEF, EFD, equal to one another: AB is parallel to CD.

For, if it be not parallel, AB and CD being produced shall meet either towards B, D, or towards A, C: Let them be produced and meet towards B, D, in the point G: Then GEF is a triangle, and its exterior angle AEF is greater than the interior and opposite angle EFG (I. 16): But it is also equal to it (*Hyp.*)—which is impossible; therefore AB and CD being produced do not meet towards B, D: And, in



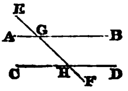
like manner it may be demonstrated that they do not meet towards A, C: But (*Def.* 35) those straight lines which, *being produced ever so far both ways, do not meet*, are *parallel* to one another: Therefore AB is parallel to CD.

Wherefore, *If a straight line &c.* Q.E.D.

PROP. XXVIII. THEOR.

If a straight line, falling upon two other straight lines, makes the exterior angle equal to the interior and opposite upon the same side of the line, or makes the interior angles upon the same side together equal to two right angles, the two straight lines shall be parallel to one another.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the exterior angle EGB equal to the interior and opposite angle GHD upon the same side: or make the interior angles on the same side BGH, GHD together equal to two right angles: AB shall be parallel to CD.



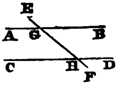
Because the angle EGB is equal to the angle GHD, and is also equal to the angle AGH (1. 15), the angle AGH is equal to the angle GHD: And they are alternate angles; therefore AB is parallel to CD (1. 27).

Again, because the angles BGH, GHD are equal to two right angles, and that AGH, BGH are also equal to two right angles (1. 13), the angles AGH, BGH are equal to the angles BGH, GHD: Take away the common angle BGH; therefore the remaining angle AGH is equal to the remaining angle GHD: And they are alternate angles; therefore AB is parallel to CD.

Wherefore, *If a straight line &c.* Q.E.D.

PROP. XXIX. THEOR.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite upon the same side, and also the two interior angles upon the same side together equal to two right angles.



Let the straight line EF fall upon the parallel straight lines AB, CD: it makes the alternate angles AGH, GHD, equal to one another, and the exterior angle EGB equal to the interior and opposite upon the same side, GHD, and the two interior angles upon the same side, BGH, GHD, together equal to two right angles.

For, if AGH be not equal to GHD, one of them must be greater than the other; let AGH be the greater: Then, because the angle AGH is greater than the angle GHD, add to each the angle BGH; therefore the angles AGH, BGH, are greater than the angles BGH, GHD: But the angles AGH, BGH, are equal to two right angles (I. 13); therefore the angles BGH, GHD, are less than two right angles: But *if a straight line falls upon two straight lines, so as to make the two interior angles on the same side of it together less than two right angles, these straight lines, being produced, shall meet* (*Ax.* 12); therefore the straight lines AB, CD, if produced far enough, shall meet—but they never meet, since they are parallel (*Hyp.*): Therefore the angle AGH is not unequal to the angle GHD, that is, it is equal to it: But the angle AGH is equal to the angle EGB (I. 15); therefore also the angle EGB is equal to the angle GHD: add to each the angle BGH; therefore the angles EGB, BGH, are equal to the angles BGH, GHD: But the angles EGB, BGH, are equal to two right angles (1. 13); therefore also the angles BGH, GHD, are equal to two right angles.

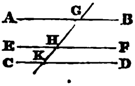
Wherefore, *If a straight line &c.* Q. E. D.

PROP. XXX. THEOR.

Straight lines, which are parallel to the same straight line, are parallel to one another.

Let AB, CD be each of them parallel to EF: AB is parallel to CD.

Let the straight line GHK cut AB, EF, CD: Then because GHK cuts the parallel straight lines AB, EF, the angle AGH is equal to the angle GHF (1. 29): Again, because the straight line GHK cuts the parallel straight lines EF, CD, the angle GHF is equal to the angle GKD (1. 29): And it was shewn that the angle AGH is equal to the angle GHF; therefore also the angle AGK is equal to the angle GKD: And they are alternate angles; therefore (1. 27) AB is parallel to CD.



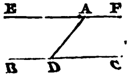
Wherefore, *Straight lines &c.* Q. E. D.

PROP. XXXI. PROB.

Through a given point to draw a straight line parallel to a given straight line.

Let A be the given point, and BC the given straight line: it is required to draw through the point A a straight line parallel to BC.

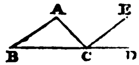
In BC take any point D, and join AD; and at the point A, in the straight line AD, make the angle DAE equal to the angle ADC; and produce the straight line EA to F.



Because the straight line AD, falling upon the two straight lines EF, BC, makes the alternate angles EAD, ADC equal to one another, therefore EF is parallel to BC (I. 27): Wherefore the straight line EAF is drawn through the given point A parallel to the given straight line BC. Q.E.F.

PROP. XXXII. THEOR.

If any side of a triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.



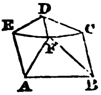
Let ABC be a triangle, and let one of its sides BC be produced to D: the exterior angle ACD shall be equal to the two interior and opposite angles CAB, ABC; and the three interior angles of the triangle, BAC, ACB, CBA, are together equal to two right angles.

Through the point C draw CE parallel to the straight line AB (I. 31): Then because AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE are equal (I. 29): Again, because AB is parallel to CE, and BD falls upon them, the exterior angle DCE is equal to the interior and opposite angle CBA (I. 29): But the angle ACE was shown to be equal to the angle BAC; therefore the whole exterior angle ACD is equal to the two interior and opposite angles BAC, CBA: To each of these equals add the angle ACB; therefore the angles ACD, ACB, are equal to the three angles BAC, ACB, CBA: But the angles ACD, ACB are equal to two right angles (I. 13);

therefore also the angles BAC, ACB, CBA, are equal to two right angles.

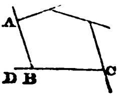
Wherefore, *If a side of a triangle &c.* Q. E. D.

Cor. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.



For any rectilineal figure ABCDE can be divided into as many triangles as the figure has sides, by drawing straight lines from any point F within the figure to each of its angles. Now, by the preceding proposition, all the angles of these triangles are equal to twice as many right angles as there are triangles, that is, as there are sides of the figure: But all the angles of the triangles make up the angles of the figure, together with the angles at the point F, that is, together with four right angles (I. 15, *Cor.* 2): Wherefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Cor. 2. All the exterior angles of any rectilineal figure are together equal to four right angles.

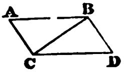


For every interior angle ABC, with its adjacent exterior ABD, is equal to two right angles (I. 13); therefore all the interior angles, together with all the exterior, are equal to twice as many right angles as there are sides of the figure, that is, by the foregoing corollary, are equal to all the interior angles, together with four right angles: Wherefore all the exterior angles are equal to four right angles.

PROP. XXXIII. THEOR.

The straight lines, which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

Let AB, CD, be equal and parallel straight lines, joined towards the same parts by the straight lines AC, BD: AC, BD, are themselves equal and parallel.



Join BC: Then because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal (1. 29): And because AB is equal to CD (*Hyp.*), and BC common to the two triangles ABC, DCB, the two sides AB, BC, are equal to the two DC, CB, and the angle ABC is equal to the angle BCD—therefore the base AC is equal to the base BD, and the triangle ABC to the triangle BCD, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle ACB is equal to the angle CBD: And because the straight line BC falling on the two straight lines AC, BD, makes the alternate angles ACB, CBD equal to one another, AC is parallel to BD (1. 27); and it has been shewn to be equal to it.

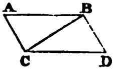
Wherefore, *Straight lines &c.* Q.E.D.

PROP. XXXIV. THEOR.

The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them, that is, divides them into two equal parts.

N.B.—A parallelogram is a four-sided figure, of which the opposite sides are parallel; and the diameter is the straight line joining two of its opposite angles.

Let ACDB be a parallelogram, of which BC is a diameter: the opposite sides and angles of the figure are equal to one another, and the diameter BC bisects it.



Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal to one another (**1.** 29); and because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD, are equal to one another: Therefore the two triangles ABC, CBD have two angles ABC, BCA in the one, equal to two angles BCD, CBD in the other, each to each, and one side BC common to the two triangles, which is adjacent to their equal angles—therefore (**1.** 26) their other sides are equal, each to each, and the third angle of the one to the third angle of the other viz. the side AB to the side CD, and AC to BD, and the angle BAC to the angle BDC: Then because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB, therefore the whole angle ABD is equal to the whole angle ACD; and the angle BAC has been shown to be equal to the angle BDC; therefore the opposite sides and angles of parallelograms are equal to one another: Also, the diameter bisects them; for AB being equal to CD, and BC common, the two AB, BC are equal to the two DC, CB, each to each, and the angle ABC is equal to the angle BCD—therefore the triangle ABC is equal to the triangle BCD, and the diameter BC divides the parallelogram ACDB into two equal parts.

Wherefore, *The opposite sides &c.* Q. E. D.

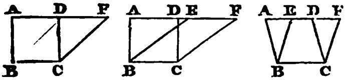
PROP. XXXV. THEOR.

Parallelograms upon the same base, and between the same parallels, are equal to one another.

Let the parallelograms ABCD, EBCF be upon the same base BC, and between the same parallels AF, BC; the parallelogram ABCD shall be equal to the parallelogram EBCF.

If (fig. 1) the points D, E, coincide, it is plain that each of the parallelograms ABCD, DBCA, is double of the triangle BDC (1. 34), and they are therefore equal to one another.

But if (fig. 2 and 3) the points D, E, do not coincide, then, because ABCD is a parallelogram, AD is equal to BC (1. 34): For the like reason EF is equal to BC; and therefore AD is equal to EF: In fig. 2 add DE to each, in fig. 3 take DE from each; then the whole, or the remainder AE, is equal to the whole, or the remainder DF: And AB is equal to DC; therefore the two EA, AB are equal to the two FD, DC, each to each, and the angle EAB is equal to the angle FDC (1. 29)—therefore the base EB is equal to the base FC, and the triangle EAB to the triangle FDC: Take the triangle FDC from the trapezium ABCF, and from the same trapezium take the triangle EAB; then the remainders are equal, that is, the parallelogram ABCD is equal to the parallelogram EBCF.

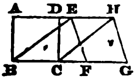


Wherefore, *Parallelograms upon the same base &c.* Q.E.D.

PROP. XXXVI. THEOR.

Parallelograms upon equal bases, and between the same parallels, are equal to one another.

Let ABCD, EFGH be parallelograms upon equal bases BC, FG, and between the same parallels AH, BG: the parallelogram ABCD shall be equal to EFGH.



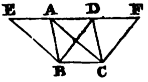
Join BE, CH: Then because BC is equal to FG, and FG to EH (1. 34), BC is equal to EH: And they are also parallels, and joined towards the same parts by the straight lines BE, CH: But *straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are themselves equal and parallel* (1. 33); therefore EB, CH are both equal and parallel, and EBCH is a parallelogram, and it is equal to ABCD, because it is upon the same base BC, and between the same parallels BC, AH (1. 35): For the like reason, the parallelogram EFGH is equal to the same EBCH: Therefore also the parallelogram ABCD is equal to EFGH.

Wherefore, *Parallelograms &c.* Q.E.D.

PROP. XXXVII. THEOR.

Triangles upon the same base, and between the same parallels, are equal to one another.

Let the triangles ABC, DBC be upon the same base BC, and between the same parallels AD, BC: the triangle ABC shall be equal to the triangle DBC.



Produce AD both ways to the points E, F, and through B draw BE parallel to CA (1. 31), and through C draw CF parallel to BD: Then each of the figures EBCA, DBCF is a parallelogram; and EBCA is equal to DBCF, because they are upon the same base BC, and between the same parallels BC, EF (1. 35): But the triangle

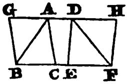
ABC is half of the parallelogram EBCA, because the diameter AB bisects it (I. 34); and the triangle DBC is half of the parallelogram DBCF, because the diameter DC bisects it: But the halves of equal things are equal (*Ax.* 7); therefore the triangle ABC is equal to the triangle DBC.

Wherefore, *Triangles &c.* Q.E.D.

PROP. XXXVIII. THEOR.

Triangles upon equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF be upon equal bases BC, EF, and between the same parallels BF, AD: the triangle ABC shall be equal to the triangle DEF.



Produce AD both ways to the points G, H; and through B draw BG parallel to CA, and through F draw FH parallel to ED: Then each of the figures GBCA, DEFH, is a parallelogram; and they are equal to one another, because they are upon equal bases BC, EF, and between the same parallels BF, GH (I. 36): But the triangle ABC is half of the parallelogram GBCA, because the diameter AB bisects it (I. 34); and the triangle DEF is half of the parallelogram DEFH, because the diameter DF bisects it: But the halves of equal things are equal; therefore the triangle ABC is equal to the triangle DEF.

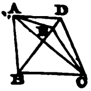
Wherefore, *Triangles &c.* Q.E.D.

PROP. XXXIX. THEOR.

Equal triangles upon the same base, and upon the same side of it, are between the same parallels.

Let the equal triangles, ABC, DBC, be upon the same base BC, and upon the same side of it: they shall be between the same parallels.

Join AD; AD is parallel to BC: For, if it is not, through the point A draw AE parallel to BC, and join EC: Then the triangle ABC is equal to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE (I. 37): But the triangle ABC is equal to the triangle DBC (*Hyp*); therefore also the triangle DBC is equal to the triangle EBC—the greater to the less, which is absurd: Therefore AE is not parallel to BC: And, in like manner, it may be demonstrated that no other line but AD can be parallel to BC; therefore AD is parallel to BC.



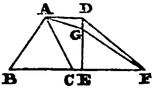
Wherefore, *Equal triangles upon &c.* Q.E.D.

PROP. XL. THEOR.

Equal triangles upon equal bases, in the same straight line, and on the same side of it, are between the same parallels.

Let the equal triangles, ABC, DEF, be upon equal bases BC, EF, in the same straight line BF, and on the same side of it: they shall be between the same parallels.

Join AD; AD is parallel to BC: For, if it is not, through A draw AG parallel to BF, and join GF: Then the triangle ABC is equal to the triangle GEF, because they are upon equal bases BC, EF, and between the same parallels BF, AG (I. 38): But the triangle ABC is equal to the triangle DEF



(*Hyp.*); therefore also the triangle DEF is equal to the triangle GEF—the greater to the less, which is absurd: Therefore AG is not parallel to BF: And, in like manner, it may be demonstrated that no other line but AD can be parallel to BF: Therefore AD is parallel to BF.

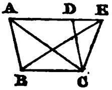
Wherefore, *Equal triangles &c.* Q.E.D.

PROP. XLI. THEOR.

If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram shall be double of the triangle.

Let the parallelogram ABCD and the triangle EBC be upon the same base BC, and between the same parallels BC, AE: the parallelogram ABCD shall be double of the triangle EBC.

Join AC: Then the triangle ABC is equal to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE (I. 37): But the parallelogram ABCD is double of the triangle ABC, because the diameter AC bisects it (I. 34); Therefore also the parallelogram ABCD is double of the triangle EBC.



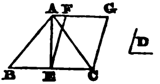
Wherefore, *If a parallelogram &c.* Q.E.D.

PROP. XLII. PROB.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle: it is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.

Bisect BC in E (I. 10); join AE, and at the point E in the straight line CE make the angle CEF equal to D (I. 23); and through A draw AG parallel to EC (I. 31), and through C draw CG parallel to EF.



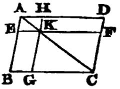
Then FECG is a parallelogram: And because BE is equal to EC, the triangle ABE is equal to the triangle AEC (I. 38), since they are upon equal bases BE, EC, and between the same parallels BC, AG; therefore the triangle ABC is double of the triangle AEC: But the parallelogram FECG is likewise double of the triangle AEC (I. 41), because they are upon the same base, and between the same parallels: Therefore the parallelogram FECG is equal to the triangle ABC (*Ax.* 6), and it has one of its angles CEF equal to the given angle D: Wherefore there has been described a parallelogram FECG equal to a given triangle ABC, and having one of its angles CEF equal to the given angle D. Q.E.F.

PROP. XLIII. THEOR.

The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.

Let ABCD be a parallelogram, of which the diameter is AC, and EH, FG the parallelograms *about* AC, that is, through which AC passes, and BK, KD the other parallelograms which make up the whole figure ABCD, and are therefore called the *complements:* the complement BK is equal to the complement KD.

For because ABCD is a parallelogram, and AC its



diameter, the triangle ABC is equal to the triangle ADC (I. 34): And, because AEKH is a parallelogram, and AK its diameter, the triangle AEK is equal to the triangle AHK: And, for the like reason, the triangle KGC is equal to the triangle KFC: Therefore, because the triangle AEK is equal to the triangle AHK, and the triangle KGC to the triangle KFC, the triangle AEK together with the triangle KGC is equal to the triangle AHK together with the triangle KFC: But the whole triangle ABC is equal to the whole triangle ADC; therefore the remainder, the complement BK, is equal to the remainder, the complement KD.

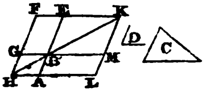
Wherefore, *The complements &c.* Q. E. D.

PROP. XLIV. PROB.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, and C the given triangle, and D the given rectilineal angle: it is required to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to D.

Make the parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D (I. 42), and place it so that BE may be in the same straight line with AB; produce FG to H, and through A draw AH parallel to BG or EF, and join HB: Then, because the straight line HF falls upon the parallels AH, EF, the angles AHF, HFE are together equal to two right angles (I. 29): Therefore the angles BHF, HFE are



together less than two right angles: But *if a straight line fall upon two straight lines, so as to make the two interior angles on the same side of it together less than two right angles, the two straight lines, being produced, shall at length meet* (*Ax.* 12); therefore HB, FE shall meet, if produced: Let them be produced to meet in K, and through K draw KL parallel to EA or FH, and produce HA, GB to the points L, M.

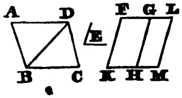
Then HLKF is a parallelogram, of which the diameter is HK; and AG, ME are the parallelograms about HK, and LB, BF are the complements; therefore LB is equal to BF (1. 43): But BF is equal to the triangle C; wherefore LB is equal to the triangle C: And because the angle GBE is equal to the angle ABM (1. 15), and likewise to the angle D (*Constr.*), therefore the angle ABM is equal to the angle D: Wherefore to the given straight line AB the parallelogram LB has been applied, equal to the triangle C, and having the angle ABM equal to the angle D. Q.E.F.

PROP. XLV. PROB.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle: it is required to describe a parallelogram equal to ABCD, and having an angle equal to E.

Join DB; and describe the parallelogram FH equal to the triangle ADB, and having the angle HKF equal to the angle E (1. 42); and to the straight line GH apply the parallelogram GM equal to the triangle DBC, and *having the angle* GHM equal to the angle E (1. 44).



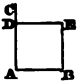
Then, because the angle E is equal to each of the angles FKH, GHM, the angle FKH is equal to GHM: To each of these equals add the angle KHG; therefore the angles FKH, KHG are equal to the angles KHG, GHM: But FKH, KHG are equal to two right angles (I. 29); therefore also KHG, GHM are equal to two right angles: And because at the point H in the straight line GH, the two straight lines KH, HM, upon opposite sides of GH, make the adjacent angles together equal to two right angles, KH is in the same straight line with HM (I. 14): Again, because the straight line GH meets the parallels FG, KM, the alternate angles FGH, GHM are equal: To each of these equals add the angle HGL; therefore the angles FGH, HGL are equal to the angles LGH, GHM: But the angles LGH, GHM are equal to two right angles (I. 29); therefore also the angles FGH, HGL are equal to two right angles, and therefore FG is in the same straight line with GL (I. 14): And because FK is parallel to GH (*Constr.*), and GH to LM, FK is parallel to LM (I. 30); and FL, KM are parallels; therefore FKML is a parallelogram: And because the parallelogram FH is equal to the triangle ABD, and the parallelogram GM to the triangle DBC, the whole parallelogram FKML is equal to the whole rectilineal figure ABCD: Wherefore the parallelogram FKML has been described equal to the given rectilineal figure ABCD, and having the angle FKM equal to the given angle E. Q.E.F.

COR. From this it is manifest, how to a given straight line to apply a parallelogram, equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle, viz. by applying to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle (I. 44); &c.

PROP. XLVI. PROB.

To describe a square upon a given straight line.

Let AB be the given straight line: it is required to describe a square upon AB.



From the point A draw AC at right angles to AB (I. 11); make AD equal to AB, and through the point D draw DE parallel to AB, and through B draw BE parallel to AD.

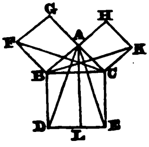
Then ADEB is a parallelogram, and therefore AB is equal to DE, and AD to BE (I. 34): But AB is equal to AD; therefore the four straight lines AB, AD, DE, BE are equal to one another, and the parallelogram ADEB is equilateral: Also all its angles are right angles: For because the straight line AD falls on the two parallels AB, DE, the angles BAD, ADE are equal to two right angles (I. 29): But BAD is a right angle (*Constr.*); therefore also ADE is a right angle: But the opposite angles of parallelograms are equal (I. 34); therefore each of the opposite angles ABE, BED is a right angle: Therefore the figure ADEB is rectangular; and it has been shewn to be equilateral; therefore it is a square (*Def.* 30), and it is described upon the given straight line AB. Q. E. F.

COR. Hence every parallelogram that has one right angle has all its angles right angles.

PROP. XLVII. THEOR.

In any right-angled triangle, the square which is described upon the side subtending the right angle is equal to the squares described upon the sides which contain the right angle.

Let ABC be a right-angled triangle having the right angle BAC . the square described upon the side BC is equal to the squares described upon BA, AC.



On BC describe the square BDEC (I. 46), and on BA, AC the squares GB, HC; and through A draw AL parallel to BD or CE, and join AD, FC.

Then, because each of the angles BAC, BAG is a right angle, the two straight lines AC, AG, upon opposite sides of AB, make with it at the point A the adjacent angles equal to two right angles; therefore CA is in the same straight line with AG (I. 14): For the same reason, AB and AH are in the same straight line: And because the angle DBC is equal to the angle FBA, each of them being a right angle, add to each the angle ABC; then the whole angle DBA is equal to the whole FBC: And because the two sides AB, BD are equal to the two FB, BC, each to each, and the angle DBA is equal to the angle FBC—therefore the base AD is equal to the base FC, and the triangle ABD to the triangle FBC: But the parallelogram BL is double of the triangle ABD, because they are upon the same base BD, and between the same parallels BD, AL (I. 41); and the square GB is double of the triangle FBC, because they are upon the same base FB, and between the same parallels FB, GC; and the doubles of equals are equal to one another; therefore the parallelogram BL is equal to the square GB: And, in like manner, by joining AE, BK, it may be demonstrated, that the parallelogram CL is equal to the square HC: Therefore the whole square BDEC is equal to the two squares GB, HC: But the square BDEC is described upon the straight line BC, and the squares

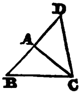
GB, HC upon BA, AC; therefore the square upon the side BC is equal to the squares upon the sides BA, AC.

Wherefore, *In any right-angled triangle &c.* Q.E.D.

PROP. XLVIII. THEOR.

If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle.

Let the square described upon BC, one of the sides of the triangle ABC, be equal to the squares upon the other sides BA, AC: the angle BAC shall be a right angle.



From the point A draw AD at right angles to AC (I. 11), and make AD equal to BA, and join DC: Then, because DA is equal to BA, the square of DA is equal to the square of BA: To each of these add the square of AC; therefore the squares of DA, AC are equal to the squares of BA, AC: But the square of DC is equal to the squares of DA, AC, because DAC is a right angle (I. 47), and the square of BC is equal to the squares of BA, AC (*Hyp.*); therefore the square of DC is equal to the square of BC, and therefore also the side DC is equal to the side BC: And because DA is equal to BA, and AC common to the two triangles DAC, BAC, the two DA, AC are equal to the two BA, AC, each to each, and the base DC is equal to the base BC—therefore the angle DAC is equal to the angle BAC (I. 8): But the angle DAC is a right angle; therefore also the angle BAC is a right angle.

Wherefore, *If the square &c.* Q.E.D.

PROBLEMS.

BOOK I.

[The more difficult Problems are distinguished by a space being left at the beginning.]

1. On a given line describe an isosceles triangle, each of whose equal sides shall be double of the base.

2. If two circles cut each other, the line joining their points of intersection is bisected at right angles by the line joining their centres.

3. From every point of a given line, the lines drawn to each of two given points on opposite sides of the line are equal: prove that the line joining the given points will bisect the given line at right angles.

4. The perpendicular is the shortest line that can be drawn from a given point to a given line, and that which is nearer to the perpendicular is less than one more remote; and from the same given point there can be drawn only two equal lines to the given line, one on each side of the shortest line.

5. From a given point without a given line draw a line making a given angle with it.

6. Construct a triangle, having given two sides and an angle opposite to one of them; and shew that there may, according to circumstances, be two solutions, one, or none.

7. Describe a circle which shall pass through two given points, and have its centre in a given line.

8. From two given points on the same side of a given line, draw two lines which shall meet in that line and make equal angles with it.

9. On a given line describe a square of which the line shall be the diagonal.

10. Take any point D in AB one of the sides of a triangle ABC, right-angled at A; in DC take DE equal to AC, and bisect CE in F: then shew that DF, BF, (drawn to a point F within the triangle ABC but *not* from the extremities of the base) are together greater than AC, BC.

11. The difference of any two sides of a triangle is less than the third side.

12. Let AB bisect CD at right angles in B; from any point E draw EC through the nearest extremity of CD cutting AB in F; and prove that the difference of EF and DF is greater than that of any other two lines drawn from E and D to meet in the line AB.

13. The sum of the diagonals of a quadrilateral is less than the sum of any four lines that can be drawn from any point whatever (except the intersection of the diagonals) to the four angles.

14. Construct a triangle, having given one side, an adjacent angle, and the sum or difference of the other two sides.

15. The centre of a circle being given, find two opposite points in the circumference by means of a pair of compasses only.

16. From a given point draw three lines of given lengths, so that their extremities may be in one line, and equally distant from each other.

17. In the figure, Euc. I. 5, if BG, CF meet in H, shew that AH bisects the angle BAC.

18. If a line which bisects the vertical angle of a triangle also bisects the base, the triangle is isosceles.

19. From a given point draw a line making equal angles with two given lines.

20. Through a given point draw a line, so that the perpendiculars upon it from two other given points, may be equal to each other.

21. The lines bisecting the angles of a triangle meet all in one point.

22. The lines bisecting at right angles the sides of a triangle meet all in one point.

23. If two sides of a triangle be produced, the lines, which bisect the two exterior angles and the third interior angle, meet all in one point.

24. If any line, joining two parallel lines, be bisected, any other line, drawn through the point of bisection to meet the two lines, will be bisected in that point.

25. The diameters of a parallelogram bisect each other.

26. A rhombus is a parallelogram, and its diagonals bisect each other at right angles.

27. The line, which joins the middle points of the oblique sides of a trapezium, is equal to the semi-sum of the two parallel sides.

28. The parallelogram, whose diameters are equal, is rectangular.

29. From a given isosceles triangle cut off a trapezium, which shall have the same base as the triangle and its remaining three sides equal to each other.

30. When the corner of a leaf of a book is twice turned down, so that the creases are parallel and the triangular fold of the same breadth as the other, shew that the space included in the second fold is three times that in the first.

31. The quadrilateral figure, whose diameters bisect each other, is a parallelogram.

32. Draw a line which would, if produced, bisect the angle between two given lines, without producing them to meet.

33. Draw a line DE, parallel to the base BC of a triangle ABC, so that DE is equal to the sum of BD and CE.

34. Draw a line DE, parallel to the base BC of a triangle ABC, so that DE is equal to the difference of BD and CE.

35. If AB be bisected in C, and from A, B, C parallel lines be drawn cutting a given line in D, E, F, shew that, according as A and B lie on the *same* or *opposite* sides of the line, CF is equal to the semi-sum or semi-difference of AD and BE.

36. Through a given point draw a line, such that the parts intercepted by it and two given lines may be equal.

37. If from the base to the sides of an isosceles triangle, three lines be drawn, making equal angles with the base, viz. one from its extremity, the other two from any point in it, these two are together equal to the first.

38. ABCD is a parallelogram; through A draw any line, and shew that the distance of C from this line is equal to the sum or difference of the distances of B and D, according as the line passes without or within the parallelogram.

39. From a given point without the angle contained by two given lines, draw a line so that the part of it between the given point and the nearest line may be equal to the part between the two given lines.

40. Of all triangles, having the same vertical angle, and whose bases pass through a given point, the least is that whose base is bisected in that point.

41. Find a point in the diagonal produced of a square, from which if a line be drawn parallel to any side of the square, and meeting another side produced, it will form, with the produced diagonal and produced side, a triangle equal to the given square.

42. In the figure, Euc. I. 5, if BG, CF meet in H, and the angles FBG, ABC be equal, then the angle BHF is double of the angle ABC.

43. If in the figure, Euc. I. 1, CA, CB be produced to meet the circumference in D, E, and F be the other point of intersection of the circles, shew that DF, EF are in one line.

44. Trisect a right angle.

45. One of the acute angles of a right-angled triangle is three times as great as the other: trisect the smaller of these.

46. If the base of an isosceles triangle be produced, twice the exterior angle is greater than two right angles by the vertical angle.

47. If there be an isosceles and an equilateral triangle upon the same base, and so that the inner vertex is equally distant from the other and from the extremities of the base, then, according as that of the former is the inner or outer vertex, its base angle will be $\frac{1}{4}$ or $2\frac{1}{2}$ times the vertical.

48. Trisect a given line: and hence divide an equilateral triangle into nine equal parts.

49. The difference of the angles at the base of any triangle is double of the angle between two lines drawn from the vertex, one bisecting the vertical angle and the other perpendicular to the base.

50. In the base BC of an isosceles triangle take any point D; in CA make CE equal to CD, and let ED cut AB in F: then shew that three times the angle AEF is greater than four right angles by the angle AFE.

51. If the sides of any polygon be produced to meet, the angles formed by these lines, together with eight right angles, are together equal to twice as many right angles as the figure has sides.

52. If the base angle of an isosceles triangle be one-fourth of the vertical angle, and from it a line be drawn perpendicular to the base to meet the opposite side produced, then the part produced, the perpendicular, and the remaining side, will form an equilateral triangle.

53. ABC is a triangle, right-angled at A, and having the angle B double of the angle C: shew that the side CB is double of the side AB.

54. If the three angles of a triangle be bisected, and one of the bisecting lines be produced to meet the opposite side, the angle contained by this line produced and one of the others is equal to the angle contained by the third line and a perpendicular from their common point of intersection to the side aforesaid.

55. If the opposite sides or opposite angles of a quadrilateral be equal, the figure will be a parallelogram.

56. If in the sides of a square, at equal distances from the four angles, four points be taken, one in each side, the figure formed by joining them will also be a square.

57. Let AD, AE be squares upon the sides of the right-angled triangle ABC, and drop DF, EG, perpendiculars on the hypothenuse BC produced: then BC and the triangle ABC are respectively equal to the sum of DF and EG, and of the triangles DBF and ECG.

58. If two opposite sides of a parallelogram be bisected, the lines drawn from the points of bisection to the opposite angles will bisect the diagonal.

59. Given the perpendicular from the vertex on the base, and the difference between each side and the adjacent segment of the base: construct the triangle.

60. Let AD be drawn perpendicular to the base BC of a triangle, in which the angle B is double of the angle C; in AB, produced or not according as the angle B is greater or less than a right angle, take BE equal to BD, and draw EDF cutting AC in F: then FA, FC, FD are equal to one another, and the triangles ABC, AFE are equiangular.

61. In [60] shew that the less side AB is equal to the sum or difference of the segments of the base, according as the greater angle B is greater or less than a right angle.

62. The lines which bisect the angles of any parallelogram, form a rectangular parallelogram, whose diameters are parallel to the sides of the former.

63. AD, BC are two parallel lines cut obliquely by AB and perpendicularly by AC; BED is drawn cutting AC in E, so that ED is equal to twice AB: prove that the angle DBC is one-third of the angle ABC.

64. In a given triangle place a line which shall be terminated by the two sides, and be equal to one given line and parallel to another.

65. Any line drawn through the bisection of the diagonal of a parallelogram to meet the sides is bisected in that point, and also bisects the parallelogram.

66. Through a given point draw a line, so that the part of it intercepted between two given parallel lines may be equal to a given line.

67. In any right-angled triangle, the middle point of the hypothenuse is equally distant from the three angles.

68. In any triangle ABC, if BE, CF be perpendiculars on any line through A, and D be the bisection of BC, shew that DE = DF.

69. If from the right angle of a triangle two lines be drawn, one bisecting the base and the other perpendicular to it, they will contain an angle equal to the difference of the two acute angles of the triangle.

70. Find the point in the base of a triangle, from which lines drawn parallel to the sides to meet them are equal.

71. The area of a trapezium is half that of a parallelogram, whose base is the sum of the two parallel sides, and altitude the perpendicular distance between them.

72. On the sides AB, AC of a triangle describe parallelograms ABDE, ACFG, and produce DE, FG to meet in H: then the area of these parallelograms together is equal to the area of the parallelogram on BC, whose side is equal and parallel to AH.

73. Upon a given base describe an isosceles triangle equal to a given triangle.

74. Shew that the perimeter of an isosceles triangle is less than that of any other equal triangle upon the same base.

75. Of all triangles having the same base and perimeter, the greatest is that which is isosceles.

76. From a given point, in one of the equal sides of an isosceles triangle, draw a line, meeting the other side produced, which shall make with these sides a triangle equal to the given triangle.

77. If one angle of a triangle be a right angle, and another be two-thirds of a right angle, shew that the equilateral triangle on the hypothenuse is equal in area to the sum of those on the sides.

78. Convert a trapezium into a triangle of equal area with one angle common; and hence shew how to transform any rectilineal figure into a triangle, whose vertex shall be in a given angle of the figure and base in one of the sides.

79. Given a triangle ABC and a point D in AB: construct another triangle ADE equal to the former, and having the common angle A.

80. Change a triangle into another equal one of given altitude.

81. In any given line, AB is taken half of AC: if through B, C, parallel lines be drawn, cutting any other line through A in D, E, then AD is half of AE, and BD of CE, and the triangle ABD a fourth of the triangle ACE; and, conversely, if BD be such that AD is half of AE, then BD is parallel to CE.

82. If the sides of any quadrilateral be bisected and the points of bisection joined, the included figure is

a parallelogram, and equal in area to half the original figure: shew also that the lines joining the bisections of opposite sides bisect each other.

83. Through D, E, the bisections of the sides AB, AC of a triangle, draw DF, EF parallel to BE, AB; and shew that the sides of the triangle DCF are equal to the three lines drawn from the angles to bisect the sides.

84. Bisect a triangle by a line drawn from a given point in one of its sides.

85. If from any point in the diagonal of a parallelogram lines be drawn to the angles, the parallelogram will be divided into two pairs of equal triangles.

86. Through E, the bisection of the diagonal BD of a quadrilateral ABCD, draw FEG parallel to AC; and shew that AG will bisect the figure.

87. ABC is a given triangle; draw BD, CE perpendicular to BC and on the same side of it, each equal to twice the altitude of the triangle; bisect AB, AC in F, G; and shew that the triangle ABC is equal to the sum or difference of the triangles BDF, CEG, according as the angles at the base of ABC are both or only one acute.

88. If of the four triangles, into which the diagonals divide a quadrilateral, two opposite ones are equal, the quadrilateral has two opposite sides parallel.

89. Upon stretching two chains AC, BD across a field, ABCD, I find that AC, BD make equal angles with CD, and that AC makes with AD the same angle that BC does with BD: hence prove that AB is parallel to CD.

90. The three lines, joining the angular points of a triangle with the middle points of the opposite sides, intersect in one point, and trisect the triangle.

91. Shew that any one of the lines in [90] is divided in the point of intersection, so that one of the parts is double of the other.

92. If two sides of a triangle be given, its area will be greatest when they contain a right angle.

93. The two triangles, formed by drawing lines from any point within a parallelogram to the extremities of two opposite sides, are together half the parallelogram.

94. If from the ends of one of the oblique sides of a trapezium two lines be drawn to the bisection of the opposite side, the triangle thus formed with the first side is half the trapezium.

95. If from the extremities of the base of an isosceles triangle lines be drawn perpendicular to the sides, the line which joins the vertex with their point of intersection will bisect the base at right angles.

96. If two exterior angles of a triangle be bisected, the line drawn from the point of intersection of the bisecting lines, to the opposite angle of the triangle, will bisect it.

97. In the figure, Euc. 1. 47, shew that, if BG and CH be joined, these lines will be parallel.

98. In ditto, if DB, EC be produced to meet FG and KH in M, N, the triangles BFM, CKN are equiangular and equal to the triangle ABC.

99. In ditto, if GH, KE, FD be joined, each of the triangles so formed is equal to the given triangle ABC.

100. In ditto, produce FG, KH to meet in M, join MB, MC, and produce MA to cut BC in L: then shew that ML is perpendicular to BC, and thence, assuming [6. 20], that the three lines AL, BK, CF intersect in one point.

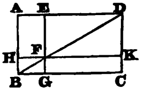
BOOK II.

DEFINITIONS.

I. **Every** right-angled parallelogram is said to be contained by any two of the straight lines which contain one of the right angles.

II. In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon.

Thus the parallelogram HG, together with the complements AF, FC, is a gnomon, which is more briefly expressed by the letters AGK, or EHC, which are at the opposite angles of the parallelograms which make the gnomon.

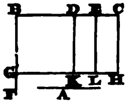


PROP. I. THEOR.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let A and BC be two straight lines, and let BC be divided into any parts in the points D, E: the rectangle contained by the straight lines A, BC shall be equal to the rectangle contained by A, BD, together with that contained by A, DE, and that contained by A, EC.

From the point B draw BF at right angles to BC, and make BG equal to A; through G draw GH parallel to BC, and through D, E, C, draw DK, EL, CH parallel to BG: Then the rectangle BH is equal to the rectangles BK, DL, EH: And BH is contained by A, BC, for it is contained by GB, BC, of which GB is equal to A; and BK is contained by A, BD, for it is contained by GB, BD, of which GB is equal to A; and DL is contained by A, DE, because DK, that is BG, is equal to A; and, in like manner, EH is contained by A, EC: Therefore the rectangle contained by A, BC, is equal to the several rectangles contained by A, BD, and by A, DE, and by A, EC.



Wherefore, *If there be two straight lines &c.* Q.E.D.

PROP. II. THEOR.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square of the whole line.

Let the straight line AB be divided into any two parts in the point C: the rectangle* AB, BC, together with the rectangle AB, AC, shall be equal to the square of AB.

Upon AB describe the square ADEB (I. 46), and

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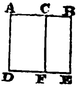
* N.B. To avoid repeating the word *contained* too frequently, the rectangle contained by two straight lines AB, BC, will in future be called, simply, the rectangle AB, BC.



through C draw CF, parallel to AD or BE: Then AE is equal to the rectangles AF, CE: And AE is the square of AB; and AF is the rectangle contained by AB, AC, for it is contained by AD, AC, of which AD is equal to AB; and CE is contained by AB, BC, for BE is equal to AB: Therefore the rectangle AB, AC, together with the rectangle AB, BC, is equal to the square of AB.

Wherefore, *If a straight line &c.* Q.E.D.

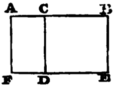
PROP. III. THEOR.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts, together with the square of the aforesaid part.

Let the straight line AB be divided into any two parts in the point C: the rectangle AB, BC shall be equal to the rectangle AC, CB, together with the square of BC.

Upon BC describe the square CDEB, and produce ED to F, and through A draw AF parallel to CD or BE:

Then the rectangle AE is equal to the rectangles AD, CE: And AE is the rectangle contained by AB, BC, for it is contained by AB, BE, of which BE is equal to BC; and AD is contained by AC, CB, for CD is equal to CB; and CE is the square of BC: Therefore the rectangle AB, BC is equal to the rectangle AC, CB, together with the square of BC.

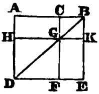


Wherefore, *If a straight line &c.* Q.E.D.

PROP. IV. THEOR.

If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let the straight line AB be divided into any two parts in C: the square of AB shall be equal to the squares of AC, CB, together with twice the rectangle contained by AC, CB.



Upon AB describe the square ADEB, and join BD, and through C draw CGF parallel to AD or BE, and through G draw HK parallel to AB or DE: Then, because CF is parallel to AD, and BD falls upon them, the exterior angle CGB is equal to the interior and opposite angle ADB (I. 29): but the angle ADB is equal to the angle ABD, because AD is equal to AB, being sides of a square; therefore the angle CGB is equal to the angle CBG, and therefore also the side CG is equal to the side CB: But CG is equal to BK, and CB to GK; therefore the figure BCGK is equilateral: It is also rectangular; for, because BK is parallel to CG, and BC meets them, the angles KBC, BCG are equal to two right angles; and KBC is a right angle, therefore also BCG is a right angle, and therefore also the angles CGK, GKB opposite to these, are right angles, and BCGK is rectangular: And it has been shown to be equilateral; therefore it is a square, and it is upon the side CB: For the like reason HF is a square, and it is upon the side HG, which is equal to AC: Therefore HF, CK are the squares of AC, CB:

And because the complement AG is equal to the complement GE (I. 43), and that AG is the rectangle contained by AC, CB, for CG is equal to CB, therefore

GE is also equal to the rectangle AC, CB, and AG, GE are equal to twice the rectangle AC, CB; and HF, CK are the squares of AC, CB; therefore the four figures HF, CK, AG, GE are together equal to the squares of AC, CB, and twice the rectangle AC, CB: But HF, CK, AG, GE make up the whole figure ADEB, which is the square of AB; therefore the square of AB is equal to the squares of AC, CB, and twice the rectangle AC, CB.

Wherefore, *If a straight line, &c.* Q.E.D.

COR. From the demonstration, it is manifest that the parallelograms about the diameter of a square are likewise squares.

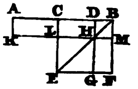
PROP. V. THEOR.

If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

Let the straight line AB be divided into two equal parts at the point C, and into two unequal parts at the point D: the rectangle AD, DB, together with the square of CD, shall be equal to the square of CB.

Upon CB describe the square CEFB, join BE, and through D draw DHG parallel to CE or BF; through H draw KLM parallel to CB or EF, and through A draw AK parallel to CL or BM.

Then, because the complement CH is equal to the complement HF, to each of these add DM; therefore the whole CM is equal to the whole DF: But CM is equal to AL, because AC is equal to CB; therefore also AL is equal to DF: To each of these add CH; therefore the whole AH is equal



to DF and CH: But AH is the rectangle contained by AD, DB, for DH is equal to DB; and DF together with CH is the gnomon CMG; therefore the rectangle AD, DB is equal to the gnomon CMG: To each of these add LG, which is equal to the square of CD (2. 4. *Cor.*); therefore the rectangle AD, DB, together with the square of CD, is equal to the gnomon CMG, together with LG: But the gnomon CMG and LG make up the whole figure CEFB, which is the square of CB; therefore the rectangle AD, DB, together with the square of CD, is equal to the square of CB.

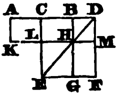
Wherefore, *If a straight line, &c.* Q. E. D.

COR. From this proposition it is manifest, that the difference of the squares of two unequal lines AC, CD, is equal to the rectangle contained by their sum and difference.

PROP. VI. THEOR.

If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to D: the rectangle AD, DB, together with the square of CB, shall be equal to the square of CD.



Upon CD describe the square CEFD, join DE, and through B draw BHG parallel to CE or DF, and through H draw KLM parallel to AD or EF, and through A draw AK parallel to CL or DM: Then, because AC is equal to CB, the rectangle AL is equal to the rectangle CH (1.36): But CH is equal to HF; therefore also AL is equal to HF: To each of these add CM; therefore the whole AM is equal to the gnomon CMG: But AM is the rectangle contained by AD, DB, for DM is equal to DB; therefore the rectangle AD, DB is equal to the gnomon CMG: To each of these add LG, which is equal to the square of CB; therefore the rectangle AD, DB, together with the square of CB, is equal to the gnomon CMG, and the figure LG: But the gnomon CMG and LG make up the whole figure CEFD, which is the square of CD; therefore the rectangle AD, DB, together with the square of CB, is equal to the square of CD.

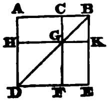
Wherefore, *If a straight line &c.* Q.E.D.

PROP. VII. THEOR.

If a straight line be divided into any two parts, the squares of the whole line and of one of the parts are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let the straight line AB be divided into any two parts at the point C: the squares of AB, BC shall be equal to twice the rectangle AB, BC, together with the square of AC.

Upon AB describe the square ADEB, and construct the figure as in the preceding propositions: Then, because AG is equal to GE, to each of these add CK; therefore the whole AK is equal to the whole CE, and AK, CE are together double of AK: But AK, CE are the gnomon AKF, together with the square CK; therefore the gnomon AKF, together with the square CK, is double of AK: But twice the rectangle AB, BC is double of AK, for BK is equal to BC; therefore the gnomon AKF, together



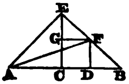
with the square CK, is equal to twice the rectangle AB, BC: To each of these equals add HF, which is equal to the square of AC; therefore the gnomon AKF, together with the squares CK, HF, is equal to twice the rectangle AB, BC, together with the square of AC: But the gnomon AKF, together with the squares CK, HF, make up the whole figure ADEB and CK, which are the squares of AB and BC; therefore the squares of AB, BC, are equal to twice the rectangle AB, BC, together with the square of AC.

PROP. VIII.

PROP. IX. THEOR.

If a straight line be divided into two equal and also into two unequal parts, the squares of the two unequal parts are together double of the square of half the line and of the square of the line between the points of section.

Let the straight line AB be divided at the point C into two equal, and at D into two unequal parts: the squares of AD, DB shall be together double of the squares of AC, CD.

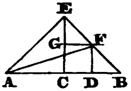


From the point C draw CE at right angles to AB, and make CE equal to CA or CB, and join EA, EB; through D draw DF parallel to CE, and through F draw FG parallel to AB; and join AF:

Then, because CA is equal to CE, the angle CAE is equal to the angle CEA; but the angle ACE is a right angle, therefore the two angles CAE, CEA are together equal to a right angle (I. 32); and they are equal to one another; therefore each of them is half a right angle: For the like reason each of the angles CBE, CEB is half a right angle: Therefore the whole angle

AEB is a right angle: And because the angle GEF is half a right angle, and EGF a right angle, for it is equal to the interior and opposite angle ECB (I. 29), the remaining angle GFE is half a right angle; therefore the angle GEF is equal to the angle GFE, and the side GE to the side GF: Again, because the angle at B is half a right angle, and FDB a right angle, (for it is equal to the interior and opposite angle ECB,) therefore the remaining angle DFB is half a right angle; therefore the angle DBF is equal to the angle DFB, and the side DB to the side DF.

Now, because AC is equal to CE, the square of AC is equal to the square of CE, and therefore the squares of AC, CE are double of the square of AC: But the square of AE is equal to the squares of AC, CE, because ACE is a right angle (I. 47); therefore the square of AE is double of the square of AC: Again, because EG is equal to GF, the square of EG is equal to the square of GF, and therefore the squares of EG, GF are double of the square of GF: But the square of EF is equal to the squares of EG, GF; therefore the square of EF is double of the square of GF, that is, of the square of CD: And the square of AE is also double of the square of AC; therefore the squares of AE, EF are double of the squares of AC, CD: And the square of AF is equal to the squares of AE, EF, because AEF is a right angle; therefore the square of AF is double of the squares of AC, CD: And the square of AF is equal to the squares of AD, DF, because ADF is a right angle; therefore the squares of AD, DF are double of the squares of AC, CD: And DF is equal to DB; therefore the squares of AD, DB are together double of the squares of AC, CD.

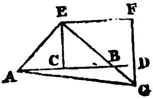


Wherefore, *If a straight line &c.* Q.E.D.

PROP. X. THEOR.

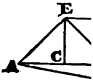
If a straight line be bisected and produced to any point, the squares of the whole line thus produced and of the part of it produced are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to D: the squares of AD, DB shall be together double of the squares of AC, CD.



From the point C draw CE at right angles to AB and make it equal to CA or CB, and join AE, EB; through E draw EF parallel to AB, and through D draw DF parallel to CE: And because the straight line EF meets the parallels EC, FD, the angles CEF, EFD are together equal to two right angles, and therefore the angles BEF, EFD are together less than two right angles: But *If a straight line meets two straight lines, so as to make the two interior angles on the same side of it together less than two right angles, these straight lines, being continually produced, shall at length meet, upon that side on which are the angles which are less thnn two right angles;* therefore EB, FD shall meet, if produced, towards B, D; let them meet in G, and join AG.

Then, because CA is equal to CE, the angle CAE is equal to the angle CEA; but the angle ACE is a right angle; therefore each of the angles CAE, CEA is half a right angle: For the like reason, each of the angles CBE, CEB is half a right angle: Therefore AEB is a right angle: And because the angle DBG is half a right angle (for it is equal to the vertically opposite angle EBC,) and that BD right angle, (for it is equal to the alternate DCE,) therefore the remaining angle DGB i a right angle, and equal to the angle DBG, and fore also the side DG is equal to the side DB: because FGE is half a right angle, and EFG a right angle, (for it is equal to the opposite angle ECD,) therefore the remaining angle FEG is half a right angle, and equal to the angle FGE, and therefore also the side FE i to the side FG.



Now, because AC is equal to CE, the square is equal to the square of CE, and therefore the s of AC, CE are double of the square of AC: B square of AE is equal to the squares of AC, CE; fore the square of AE is double of the square c Again, because EF is equal to FG, the square o equal to the square of FG, and therefore the s of EF, FG are double of the square of EF: B square of EG is equal to the squares of EF, FG; fore the square of EG is double of the square that is, of the square of CD: And the square of also double of the square of AC; therefore the s of AE, EG are double of the squares of AC, CD the square of AG is equal to the squares of AI therefore the square of AG is double of the squ AC, CD: And the square of AG is equal to the s of AD, DG; therefore the squares of AD, DG are of the squares of AC, CD: And DG is equal t therefore the squares of AD, DB are double squares of AC, CD.

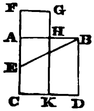
Wherefore, *If a straight line &c.*

PROP. XI. PROB.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.

Let AB be the given straight line: it is required to divide it into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.

Upon AB describe the square ACDB; bisect AC in E, and join BE; produce CA to F, and make EF equal to EB; and upon AF describe the square AFGH: AB is divided in H, so that the rectangle AB, BH is equal to the square of AH.



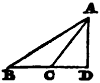
Produce GH to K: Then, because the straight line AC is bisected in E, and produced to F, the rectangle CF, FA, together with the square of AE, is equal to the square of EF (2. 6): But EF is equal to EB; therefore the rectangle CF, FA, together with the square of AE, is equal to the square of EB: And the square of EB is equal to the squares of AE, AB, because the angle EAB is a right angle; therefore the rectangle CF, FA, together with the square of AE, is equal to the squares of AE, AB: Take away the common square of AE; therefore the remainder, the rectangle CF, FA, is equal to the square of AB: And the figure FK is the rectangle contained by CF, FA, for FG is equal to FA, and AD is the square of AB; therefore FK is equal to AD: Take away the common part AK, and the remainder FH is equal to the remainder HD: But HD is the rectangle contained by AB, BH, (for AB is equal to BD,) and

FH is the square of AH; therefore the rectangle AB, BH is equal to the square of AH: Wherefore the given straight line AB is divided in H, so that the rectangle AB, BH is equal to the square of AH. Q. E. F.

PROP. XII. THEOR.

In obtuse-angled triangles, if a perpendicular be drawn from any of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted, without the triangle, between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled triangle, having the obtuse angle ACB, and from the point A let AD be drawn perpendicular to BC produced: the square of AB is greater than the squares of AC, CB, by twice the rectangle BC, CD.



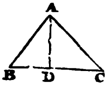
Because the straight line BD is divided into two parts in the point C, the square of BD is equal to the squares of BC, CD, and twice the rectangle BC, CD (2. 4): To each of these equals add the square of DA; therefore the squares of BD, DA are equal to the squares of BC, CD, DA, and twice the rectangle BC, CD: But the square of BA is equal to the squares of BD, DA, because the angle at D is a right angle, and the square of CA is equal to the squares of CD, DA; therefore the square of BA is equal to the squares of BC, CA, and twice the rectangle BC, CD, that is, the square of BA is greater than the squares of BC, CA, by twice the rectangle BC, CD.

Wherefore, *If obtuse-angled triangles &c.* Q. E. D.

PROP. XIII. THEOR.

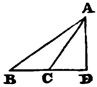
In every triangle, the square of the side subtending any of the acute angles is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle and the acute angle.

Let ABC be any triangle, and the angle at B one of its acute angles; and upon BC, one of the sides containing it, let fall the perpendicular AD from the opposite angle: the square of AC, opposite to the angle B, is less than the squares of CB, BA, by twice the rectangle CB, BD.

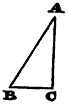


First, let AD fall within the triangle ABC: Then, because the straight line CB is divided into two parts in the point D, the squares of CB, BD are equal to twice the rectangle CB, BD, and the square of CD (2. 7): To each of these equals add the square of DA; therefore the squares of CB, BD, DA are equal to twice the rectangle CB, BD, and the squares of CD, DA: But the square of AB is equal to the squares of BD, DA, (because the angle BDA is a right angle,) and the square of AC is equal to the squares of CD, DA; therefore the squares of CB, BA are equal to the square of AC, and twice the rectangle CB, BD, that is, the square of AC alone is less than the squares of CB, BA, by twice the rectangle CB, BD.

Next, let AD fall without the triangle ABC: Then, because the angle at D is a right angle, the angle ACB is greater than a right angle (**1.** 16); and therefore the square of AB is equal to the squares of AC, CB, and twice the rectangle BC, CD (**2.** 12): To each of these equals add the square of BC; therefore the squares of AB, BC are equal to the square of AC, and twice the rectangle BC, CD, and twice the square of BC: But because BD is divided into two parts in C, the rectangle DB, BC is equal to the rectangle BC, CD, and the square of BC (**2.** 3); therefore twice the rectangle DB, BC is equal to twice the rectangle BC, CD, and twice the square of BC; and therefore the squares of AB, BC are equal to the square of AC, and twice the rectangle DB, BC, that is, the square of AC alone is less than the squares of AB, BC, by twice the rectangle DB, BC.



Lastly, let the side AC be perpendicular to BC: Then BC is the straight line between the perpendicular and the acute angle at B; and it is manifest, that the squares of AB, BC, are equal to the square of AC and twice the square of BC (**1.** 47).



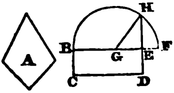
Wherefore, *In every triangle &c.* Q.E.D.

PROP. XIV. PROB.

To describe a square that shall be equal to a given rectilineal figure.

Let A be the given rectilineal figure: it is required to describe a square that shall be equal to A.

Describe (**1.** 45) a rectangular parallelogram BCDE equal to the rectilineal figure A: If then the sides of it, BE, ED are equal to one another, it is a square, and what was required is now done: But if they are not equal, produce one of them BE to F, and make EF equal to ED, and bisect BF in G; and from the centre G, at the distance GB, or GF, describe the semicircle BHF, and produce DE to H, and join GH.



Then, because the straight line BF is divided into two equal parts at G and into two unequal at E, the rectangle BE, EF, together with the square of GE, is equal to the square of GF (2. 5): But GF is equal to GH; therefore the rectangle BE, EF, together with the square of GE, is equal to the square of GH: But the square of GH is equal to the squares of GE, EH; therefore the rectangle BE, EF, together with the square of GE, is equal to the squares of GE, EH: Take away the square of GE, which is common to both; and the remainder, the rectangle BE, EF, is equal to the remainder, the square of EH: But the rectangle contained by BE, EF is the parallelogram BD, because EF is equal to ED; therefore BD is equal to the square of EH: But BD is equal to the rectilineal figure A; therefore the square of EH is equal to the rectilineal figure A: Wherefore a square has been made equal to the given rectilineal figure A, viz. the square described upon EH. Q. E. F.

PROBLEMS.

BOOK II.

1. If a line be drawn from one of the acute angles of a right-angled triangle to the bisection of the opposite side, the square upon that line is less than the square upon the hypothenuse by three times the square upon half the line bisected.

2. If from the middle point of one of the sides of a right-angled triangle a perpendicular be drawn to the hypothenuse, the difference of the squares of the segments so formed is equal to the square of the other side.

3. In any triangle, if a perpendicular be drawn from the vertex to the base, the difference of the squares upon the sides is equal to the difference of the squares upon the segments of the base.

4. Let AOB be a quadrant of a circle, whose centre is O; from any point C in its arc draw CD perpendicular to OA or OB, meeting in E the radius which bisects the angle AOB: then shew that the squares upon CD, DE are together equal to the square upon OA.

5. If from any point in the diameter of a semicircle two lines be drawn to the circumference, one to the bisection of the arc, and the other perpendicular to the diameter, then the squares upon these two lines are together double of the square upon the radius.

6. If A be the vertex of an isosceles triangle ABC, and CD be drawn perpendicular to AB, prove that the squares upon the three sides are together equal to the square on BD and twice the square on AD and thrice the square on CD.

7. If from any point perpendiculars be dropped on all the sides of any rectilineal figure, the sum of the squares upon the alternate segments of the sides will be equal.

8. If from one of the acute angles of a right-angled triangle a line be drawn to the opposite side, the squares of that side and the line so drawn are together equal to the squares of the hypothenuse and the segment adjacent to the right angle.

9. Describe a square equal to the difference of two given squares.

10. Divide, when possible, a given line into two parts, so that the sum of their squares may be equal to a given square.

11. From D the middle point of AC, one of the sides of an equilateral triangle ABC, draw DE perpendicular on BC; and shew that the square upon BD is three-fourths of the square upon BC, and the line BE three-fourths of BC.

12. If from the vertex A, of a right-angled triangle BAC, AD be dropped perpendicular on the base, shew that the rectangles of BC and BD, BC and CD, BD and CD, are respectively equal to the squares upon AB, AC, AD.

13. Produce a given line so that the rectangle of the whole line produced and the original line shall be equal to a given square.

14. If on the radius of a circle a semicircle be described, and a perpendicular to the common diameter be drawn, the square of the chord of the greater circle, between the extremity of the diameter and the point of section of the perpendicular, will be double of the square of the corresponding chord of the lesser circle.

15. Divide a line in two points equally distant from its extremities, so that the square on the middle part shall be equal to the sum of the squares on the extremes: and shew also that in this case the square

of the whole line will be equal to the squares of the extreme parts together with twice the rectangle of the whole and the middle part.

16. Divide a line into two parts, so that the squares of the whole line and one of the parts shall be together double of the square of the other part: and shew that, by the same division, the square of the greater part will be equal to twice the rectangle of the whole and the lesser part.

17. Divide a straight line into two parts so that the sum of their squares may be the least possible.

18. Shew that the sum of the squares upon two lines is never less than twice their rectangle, and that the difference of their squares is equal to the rectangle of their sum and difference.

19. Shew that of the two algebraical expressions,

$$(a+x)\ (a-x)+x^2=a^2,\ \ (a+x)^2+(a-x)^2=2a^2+2x^2,$$

the first is equivalent to Props. V. and VI., and the second to Props. IX. and X., of Euc. 2.

20. ABCD is a rectangle, E any point in BC, F in CD: shew that the rectangle ABCD is equal to twice the triangle AEF together with the rectangle BE, DF.

21. If a line be divided into two equal and also into two unequal parts, the squares of the two unequal parts are together equal to twice the rectangle contained by these parts together with four times the square of the line between the points of section.

22. If from one of the equal angles of an isosceles triangle a perpendicular be dropped on the opposite side, the rectangle of that side and the segment of it between the perpendicular and base is equal to half the square upon the base.

23. A, B, C, D, are four points in the same line, E a point in that line equally distant from the middle of the segments AB, CD, F any other point in AD: shew that the squares of AF, BF, CF, DF, are together greater than the squares of AE, BE, CE, DE, by four times the square of EF.

24. If from the extremities of any chord in a circle lines be drawn to any point in the diameter to which it is parallel, the sum of their squares is equal to the sum of the squares upon the segments of the diameter.

25. If the sides of a triangle be as 2, 4, 5, shew whether it will be acute- or obtuse-angled.

26. In any isosceles triangle ABC, if AD be drawn from the vertex to any point in the base, shew that the difference of the squares on AB and AD is equal to the rectangle of BD and CD.

27. If in the figure, Euc. I. 47, the angular points be joined, the sum of the squares of the six sides of the figure so formed is equal to eight times the square of the hypothenuse.

28. If one angle of a triangle be four-thirds of a right angle, the square of the side subtending that angle is equal to the sum of the squares of the sides containing it, together with the rectangle contained by these sides.

29. If ABC be a triangle, with the angles at B, C, each double of the angle at A, then the square of AB is equal to the square of BC, together with the rectangle of AB and BC.

30. In any triangle ABC, if BP, CQ be drawn perpendicular to AC, AB, produced if necessary, then shall the square of BC be equal to the rectangle of AB, BQ, together with the rectangle of AC, CP.

31. In [22] shew that the square of the perpendicular is equal to the square of the line between the perpendicular and the other equal angle, together with twice the rectangle of the segments of the side.

32. If from the right angle of a right-angled triangle lines be drawn to the opposite angles of the square described on the hypothenuse, the difference of the squares on these lines is equal to the difference of the squares on the two sides of the triangle.

33. In any triangle the squares of the two sides are together double of the squares of half the base, and of the line joining its middle point with the opposite angle.

34. If BD be drawn bisecting AC, one of the sides of the triangle ABC, in D, and AE be drawn perpendicular to the base, shew that the square upon BD is equal to the sum or difference of the square upon the half of BD and the rectangle BC, BE, according as E lies in BC or in BC produced.

35. Any rectangle is half the rectangle contained by the diameters of the squares upon its two sides.

36. If from any point within a rectangle lines be drawn to the angular points, the sums of the squares upon those drawn to the opposite angles will be equal.

37. The squares of the diagonals of a parallelogram are together equal to the squares of the four sides.

38. The squares of the diagonals of a quadrilateral are together less than the squares of the four sides by four times the square of the line joining the bisections of the diagonals.

39. The squares of the diagonals of any quadrilateral are together double of the squares of the two lines joining the bisections of the opposite sides.

40. The squares of the sides of any triangle are together triple of the squares of the distances of the angles from the point of intersection of lines drawn from them to the bisections of the opposite sides.

41. If two opposite sides of any quadrilateral be bisected, the sum of the squares of the other two sides, together with the squares of the diagonals, is equal to the sum of the squares of the sides bisected together with four times the square of the line joining the points of section.

42. If DE be drawn parallel to the base BC of an isosceles triangle ABC, then the square of BE is equal to the rectangle of BC, CD, together with the square of CE.

43. The squares of the diagonals of a trapezium are together equal to the squares of its two parallel sides, with twice the rectangle contained by its parallel sides.

44. If BD, CE be squares described upon the sides AB, AC, of any triangle, shew that the squares of BC and DE are together double of the squares of AB and AC.

45. If squares be described on the three sides of any triangle, and the angular points of the squares be joined, the sum of the squares of the sides of the hexagonal figure thus formed will be equal to four times the sum of the squares of the sides of the triangle.

46. If two points be taken in the diameter of a circle equally distant from the centre, the sum of the squares of two lines drawn from these points to any point in the circumference will be constant.

47. The hypothenuse AB of a right-angled triangle ABC is trisected in the points D, E: shew that, if CD, CE be joined, the sum of the squares on the sides of the triangle CDE is equal to two-thirds of the square on AB.

48. Divide a given line into two parts, so that their rectangle may be equal to a given square.

49. If the areas of a triangle and of a square be equal, the perimeter of the triangle will be the greater.

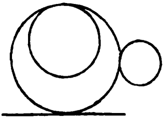
50. ABCD is a quadrilateral, E the middle point of the line joining the bisections of the diagonals; if with E as centre any circle be described, shew that for every point P in this circle, $PA^2+PB^2+PC^2+PD^2$ is constant, and equals $EA^2+EB^2+EC^2+ED^2+4EP^2$.

BOOK III.

DEFINITIONS.

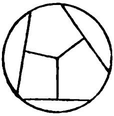
I. Equal circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.

This is not a definition, but a theorem, the truth of which is evident; for, if the circles be applied to one another so that their centres coincide, the circles must likewise coincide, since the straight lines from the centres are equal.



II. A straight line is said to touch a circle, when it meets the circle, and being produced does not cut it.

III. Circles are said to touch one another, which meet but do not cut one another.



IV. Straight lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal:

V. And the straight line on which the greater perpendicular falls, is said to be farther from the centre.

VI. A segment of a circle is the figure contained by a straight line and the circumference it cuts off.

VII. The angle of a segment is that which is contained by the straight line and the circumference.

VIII. An angle in a segment is the angle contained *by two* straight lines drawn from any point in the circumference of the segment to the extremities of the straight line which is the base of the segment:

IX. And an angle is said to insist or stand upon the circumference intercepted between the straight lines that contain the angle.

X. A sector of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.

XI. Similar segments of circles are those in which the angles are equal, or which contain equal angles.

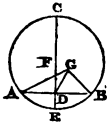
PROP. I. PROB.

To find the centre of a given circle.

Let ABC be the given circle: it is required to find its centre.

Draw within it any straight line AB and bisect it in D; from the point D draw DC at right angles to AB, and produce CD to E, and bisect CE in F: the point F shall be the centre of the circle ABC.

For, if the centre be in CE, it is plain that it must be F, the middle point of CE: but if it be not in CE, let, if possible, G be the centre, and join GA, GD, GB: Then, because AD is equal to BD, and DG common to the two triangles ADG, BDG, the two sides AD, DG are equal to the two BD, DG, each to each, and the base AG is equal to the base BG, because they are both drawn from the centre G to the circumference—therefore the angle ADG is equal to the angle BDG: But *when a straight line, standing upon another straight line, makes the adjacent*



angles equal to one another, each of these angles is a right angle; therefore the angle BDG is a right angle: But BDF is also a right angle; therefore the angle BDF is equal to the angle BDG, the greater to the less—which is absurd: therefore G is not the centre of the circle ABC: And in like manner it can be shewn that no other point but F is the centre; that is, F has been found, the centre of the circle ABC. Q.E.F.

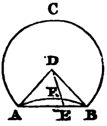
COR. From this it is manifest that, if in a circle one straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

PROP. II. THEOR.

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

Let ABC be a circle, and A, B any two points in the circumference: the straight line drawn from A to B shall fall within the circle.

For if not, let it fall, if possible, without, as AEB: find D the centre of the circle ABC; and join DA, DB; in the circumference AB take any point F, join DF, and produce it to E: Then, because DA is equal to DB, the angle DAB is equal to the angle DBA: And because AE, a side of the triangle DAE, is produced to B, the exterior angle DEB is greater than the angle DAE: But DAE was proved equal to the angle DBE; therefore the angle DEB is



N.B.—Whenever the expression "straight lines from the centre," or "drawn from the centre," occurs, it is to be understood that they are drawn to the circumference.

greater than the angle DBE: But *to the greater angle the greater side is opposite;* therefore DB is greater than DE: But DB is equal to DF; therefore DF is greater than DE, the less than the greater—which is absurd; therefore the straight line drawn from A to B does not fall without the circle: And, in like manner, it may be demonstrated that it does not fall upon the circumference; therefore it falls within it.

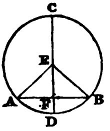
Wherefore, *If any two points &c.* Q.E.D.

PROP. III. THEOR.

If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and, if it cuts it at right angles, it shall bisect it.

Let ABC be a circle, and let CD, a straight line drawn through the centre, bisect any straight line AB, which does not pass through the centre, in the point F: it shall cut AB at right angles.

Take E the centre of the circle, and join EA, EB: Then, because AF is equal to BF, and FE common to the two triangles AFE, BFE, the two sides AF, FE, of the one are equal to the two, BF, FE, of the other, each to each; and the base AE is equal to the base BE—therefore the angle AFE is equal to the angle BFE: But *when a straight line, standing upon another straight line, makes the adjacent angles equal to one another, each of them is a right angle;* therefore each of the angles AFE, BFE is a right angle; and therefore the straight line CD, drawn through the centre, bisecting another AB that does not pass through the centre, cuts the same at right angles.



Next, let CD cut AB at right angles: CD shall also bisect it, that is, AF shall be equal to FB.

The same construction being made, because EA, EB, drawn from the centre, are equal to one another, the angle EAF is equal to the angle EBF: And the right angle AFE is equal to the right angle BFE; therefore in the two triangles EAF, EBF, there are two angles in the one equal to two angles in the other, each to each, and the side EF, which is opposite to one of the equal angles in each, is common to both—therefore the other sides are equal, and therefore AF is equal to FB.

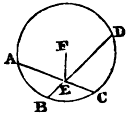
Wherefore, *If a straight line &c.* Q.E.D.

PROP. IV. THEOR.

If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect each other.

Let ABCD be a circle, and AC, BD two straight lines in it, which cut one another in the point E, and do not both pass through the centre: AC, BD, shall not bisect each other.

If one of the lines pass through the centre, it is plain that it cannot be bisected by the other which does not pass through the centre: But if neither of them pass through the centre, let, if possible, AE be equal to EC, and BE to ED; and take F the centre of the circle, and join FE: Then, because FE, a straight line through the centre, bisects another AC which does not pass through the centre, it cuts it at right angles (3. 3); therefore FEA is a right angle: Again, because the straight line FE bisects the straight line BD, which does not pass through the



centre, it cuts it at right angles; therefore also FEB is a right angle: but FEA was shown to be a right angle; therefore the angle FEA is equal to the angle FEB, the less to the greater—which is absurd: Therefore AC, BD do not bisect one another.

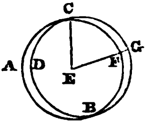
Wherefore, *If in a circle &c.* Q.E.D.

PROP. V. THEOR.

If two circles cut one another, they shall not have the same centre.

Let the two circles ABC, CDG cut one another in the points B, C: they shall not have the same centre.

For, if possible, let E be their centre; join EC, and draw any straight line EFG cutting the circles in F and G: Then, because E is the centre of the circle ABC, EC is equal to EF: Again, because E is the centre of the circle CDG, EC is equal to EG: But EC was shown to be equal to EF; therefore EF is equal to EG, the less to the greater—which is absurd: Therefore E is not the centre of the circles ABC, CDG.



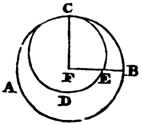
Wherefore, *If two circles &c.* Q.E.D.

PROP. VI. THEOR.

If two circles touch one another internally, they shall not have the same centre.

Let the two circles ABC, CDE, touch one another internally in the point C: they shall not have the same centre.

For, if possible, let F be their centre: join FC, and draw any straight line FEB, cutting the circles in E and B: Then, because F is the centre of the circle ABC, FC is equal to FB: Again, because F is the centre of the circle CDE, FC is equal to FE: But FC was shewn to be equal to FB; therefore FE is equal to FB, the less to the greater—which is absurd: therefore F is not the centre of the circles ABC, CDE.



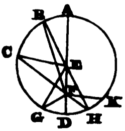
Wherefore, *If two circles &c.* Q. E. D.

PROP. VII. THEOR.

If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of the diameter is the least; and, of any others, that which is nearer to the line which passes through the centre is always greater than one more remote: and from the same point there can be drawn two straight lines, and only two, that are equal to one another, one upon each side of the shortest line.

Let ABCD be a circle, and AD its diameter, in which let any point F be taken which is not the centre, and let the centre be E: of all the straight lines FB, FC, FG, &c., that can be drawn from F to the circumference, FA, which passes through E, shall be the greatest, and FD, the other part of the diameter AD, shall be the least: and of the others, FB shall be greater than FC, and FC than FG.

Join BE, CE, GE: Then, because any two sides of a triangle are greater than the third, therefore BE, EF are to-



gether greater than BF: but BE is equal to AE; therefore AE, EF, that is AF, is greater than BF: Again, because BE is equal to CE, and EF common to the triangles BEF, CEF, the two sides BE, EF are equal to the two CE, EF: but the angle BEF is greater than the angle CEF; therefore (I. 24) the base BF is greater than the base CF: And, in like manner, it may be shewn that CF is greater than GF: Again, because EF, FG are greater than EG, and that EG is equal to ED, therefore EF, FG are greater than ED; take away the common part EF, and the remainder FG is greater than the remainder FD: Therefore AF is the greatest, and FD the least, of all the straight lines from F to the circumference, and FB is greater than FC, and FC than FG.

Also there can be drawn two equal straight lines from the point F to the circumference, one upon each side of the shortest line FD: For, at the point E, in the straight line EF, make the angle FEH equal to the angle FEG, and join FH: Then, because GE is equal to HE, and EF common to the two triangles GEF, HEF, the two sides GE, EF are equal to the two HE, EF, each to each, and the angle GEF is equal to the angle HEF—therefore the base FG is equal to the base FH.

But, besides FH, no other straight line can be drawn from F to the circumference, equal to FG: For, if there can, let it be FK: Then, because FK is equal to FG, and FG to FH, therefore also FK is equal to FH, that is, a line nearer to that which passes through the centre, is equal to one which is more remote—which is impossible.

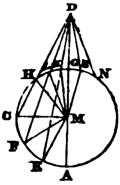
Wherefore, *If any point &c.* Q. E. D.

PROP. VIII. THEOR.

If any point be taken without a circle, from which straight lines are drawn to the circumference, whereof one passes through the centre, of those which fall upon the concave circumference, the greatest is that which passes through the centre, and of the rest, that which is nearer to the one passing through the centre is greater than one more remote; and of those which fall upon the convex circumference, the least is that between the point without the circle and the diameter, and of the rest, that which is nearer to the least is less than one more remote: and only two equal straight lines can be drawn from the given point to the circumference, one upon each side of the shortest line.

Let ABC be a circle, and D any point without it, from which let the straight lines DA, DE, DF, DC be drawn to the circumference, whereof DA passes through the centre: of those which fall upon the concave circumference AEFC, the greatest is DA which passes through the centre, and that which is nearer to it is greater than one more remote, viz. DE than DF, and DF than DC; and of those which fall upon the convex circumference GKLH, the least is DG, between the point D and the diameter AG, and that which is nearer to it is less than one more remote, viz. DK than DL, and DL than DH.

Take M the centre of the circle ABC (**3**. 1), and join ME, MF, MC, MH, ML, MK: Then because any two sides of a triangle are greater than the third side, therefore EM, MD are together greater than ED: but EM is equal to AM; therefore AM, MD, that is, AD, is greater than ED: Again, because EM is equal to FM, and MD common to the triangles EMD, FMD, the two sides EM, MD are equal to the two FM, MD: but the angle EMD is greater than the angle FMD; therefore (I. 24) the base ED is greater than the base FD: And, in like manner, it may be shewn that FD is greater than CD: Therefore DA is the greatest, and DE greater than DF, and DF than DC.



Again, because MK, KD are greater than MD, and MK is equal to MG, the remainder KD is greater than the remainder GD, that is, GD is less than KD: And because MK, KD are drawn to the point K within the triangle MLD, from M, D, the extremities of its side MD, therefore MK, KD are less than ML, LD (I. 21): and MK is equal to ML; therefore the remainder KD is less than the remainder LD: And, in like manner, it may be shewn that LD is less than HD: Therefore DG is the least, and DK less than DL, and DL than DH.

Also, there can be drawn two equal straight lines from the point D to the circumference, one upon each side of the shortest line DG: For, at the point M, in the straight line MD, make the angle DMB equal to the angle DMK, and join DB: Then, because MK is equal to MB, and DM common to the triangles DMK, DMB, the two sides DM, MK are equal to the two DM, MB, and the angle DMK is equal to the angle DMB—therefore the base DK is equal to the base DB.

But, besides DB, no other straight line can be drawn from D to the circumference, equal to DK: For, if there can, let it be DN: Then, because DN is equal to DK, and DK to DB, therefore also DN is equal to DB.

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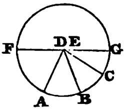
that is, a line, nearer to the least line, is equal to one which is more remote—which is impossible.

Wherefore, *If any point &c.* Q.E.D.

PROP. IX. THEOR.

If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC, from which to the circumference there fall more than two equal straight lines, viz. DA, DB, DC: the point D is the centre of the circle.



For, if not, let E be the centre; join DE, and produce it both ways to meet the circumference in F and G.

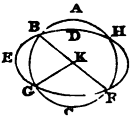
Then FG is a diameter of the circle ABC: And, because in FG, a diameter of the circle ABC, there is taken the point D, which is not the centre, DG shall be the greatest line from D to the circumference, and DC greater than DB, and DB than DA (3. 7): But they are likewise equal (*Hyp.*)—which is absurd: Therefore E is not the centre of the circle ABC: And, in like manner, it may be demonstrated, that any other point than D is not the centre: Therefore D is the centre of the circle ABC.

Wherefore, *If a point &c.* Q.E.D.

PROP. X. THEOR.

One circumference of a circle cannot cut another in more than two points.

If it be possible, let the circumference ABF cut the circumference DBF in more than two points, viz. in points B, G, F.

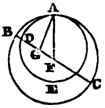


Take the centre K of the circle ABF, and join KB, KG, KF: Then, because within the circle DBF there is taken the point K, from which to the circumference DBF fall more than two equal straight lines KB, KG, KF, the point K (3. 9) is the centre of the circle DBF: But K is also the centre of the circle ABF; therefore the same point is the centre of two circles that cut one another—which is impossible (3. 5).

Wherefore, *One circumference &c.* Q.E.D.

PROP. XI. THEOR.

If two circles touch each other internally, the straight line which joins their centres, being produced, shall pass through the point of contact.



Let the two circles ABC, ADE touch each other internally in the point A; and let F be the centre of the circle ABC, and G the centre of the circle ADE: the straight line which joins the centres F, G, being produced, shall pass through the point of contact, A.

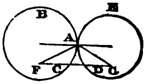
For, if not, let it fall otherwise, if possible, as FGDB, and join AF, AG: Then, because AG, GF are greater than AF, that is, than BF, (for AF is equal to BF, both being from the same centre,) take away the common part FG; therefore the remainder AG is greater than the remainder BG: But AG is equal to DG; therefore DG is greater than BG, the less than the greater—which is absurd: Therefore the straight line which joins the centres F, G cannot pass otherwise than through the point of contact A, that is, it must pass through it.

Wherefore, *If two circles &c.* Q.E.D.

PROP. XII. THEOR.

If two circles touch each other externally, the straight line which joins their centres shall pass through the point of contact.

Let the two circles ABC, ADE touch each other externally in the point A, and let F be the centre of ABC, and G the centre of ADE: the straight line which joins the points F, G, shall pass through the point of contact A.



For, if not, let it pass otherwise, if possible, as FCDG, and join AF, AG: Then, because F is the centre of the circle ABC, FA is equal to FC; and, because G is the centre of the circle ADE, GA is equal to GD: Therefore FA, AG are together equal to FC, DG, and therefore they are together less than the whole FG: But they are also together greater than it (I. 20)—which is impossible: Therefore the straight line which joins the centres F, G, cannot pass otherwise than through the point of contact A, that is, it must pass through it.

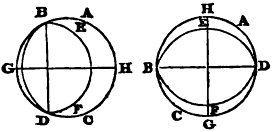
Wherefore, *If two circles &c.* Q.E.D.

PROP. XIII. THEOR.

One circle cannot touch another in more points than one, whether it touches it on the inside or outside.

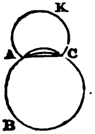
For, if it be possible, let the circle EBF touch the circle ABC in more points than one, and first on the inside, in the points B, D.

Join BD, and draw GH bisecting BD at right angles: Then, because the points B, D are in the circumference of each of the circles, the straight line BD falls within each of them (3. 2); and therefore the centres of each (3. 1. Cor.) must be in the straight line GH which bisects BD at right angles; and therefore also GH must pass through the point of contact (3. 11): But it does not pass through it, because the points B, D are not in the straight line GH—which is absurd: Therefore one circle cannot touch another on the inside in more points than one.



Nor can two circles touch one another on the outside in more than one point.

For, if it be possible, let the circle ACK touch the circle ABC in the points A, C, and join AC: Then, because the two points A, C, are in the circumference of the circle ACK, the straight line AC which joins them (3. 2) must fall within the circle ACK: And the circle ACK is without the circle ABC; therefore also the straight line AC is without the circle ABC: But because the two points A, C, are in the circumference of the circle ABC, the straight line AC falls within the circle ABC—which is absurd: Therefore one circle cannot touch another on the outside in more than one point: And it has been shewn, that they cannot touch on the inside in more points than one.



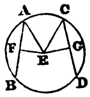
Wherefore, *One circle &c.* Q. E. D.

PROP. XIV. THEOR.

Equal straight lines in a circle are equally distant from the centre; and, conversely, those, which are equally distant from the centre, are equal to one another.

Let the straight lines AB, CD, in the circle ABDC, be equal to one another: they shall be equally distant from the centre.

Take E, the centre of the circle ABDC, and from it draw EF, EG, perpendiculars to AB, CD: Then, because the straight line EF, passing through the centre, cuts the straight line AB, which does not pass through the centre, at right angles, it also bisects it (3. 3); and therefore AF is equal to FB, and AB is double of AF: And, in like manner, it may be shewn that CD is double of CG: But AB is equal to CD (*Hyp.*); and therefore also AF is equal to CG.



Now, because AE is equal to EC, the square of AE is equal to the square of CE: But the squares of AF, FE are equal to the square of AE, because the angle AFE is a right angle, and for the like reason, the squares of CG, GE are equal to the square of CE; therefore the squares of AF, FE are equal to the squares of CG, GE: But the square of AF is equal to the square of CG, because AF is equal to CG; therefore the remaining square of FE is equal to the remaining square of GE, and the straight line EF is therefore equal to EG: But straight lines in a circle are said to be equally distant from the centre, when the perpendiculars drawn to them from the centre are equal (3. *Def.* 4); therefore AB, CD are equally distant from the centre.

Next, let the straight lines AB, CD be equally distant from the centre, that is, let EF be equal to EG: then AB shall be equal to CD.

For, the same construction being made, it may be shewn, as before, that AB is double of AF, and CD of CG, and that the squares of AF, FE are equal to the squares of CG, GE, of which the square of FE is equal to the square of GE, because FE is equal to GE; therefore the remaining square of AF is equal to the remaining square of CG, and the line AF to the line CG: But AB is double of AF, and CD of CG; therefore AB is equal to CD.

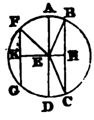
Wherefore, *Equal straight lines &c.* Q. E. D.

PROP. XV. THEOR.

The diameter is the greatest straight line in a circle, and, of all others, that which is nearer to the centre is greater than one more remote: and, conversely, the greater is nearer to the centre than the less.

Let ABCD be a circle, of which the diameter is AD, and the centre E, and let BC be nearer to the centre than FG: AD shall be greater than any straight line BC, which is not a diameter, and BC than FG.

From the centre draw EH, EK, perpendiculars to BC, FG, and join EB, EC, EF: Then, because AE is equal to BE, and ED to EC, therefore the whole AD is equal to the two BE, EC: But BE, EC, are greater than BC; therefore also AD is greater than BC: Again, because BC is nearer to the centre than FG, therefore EH is less than EK (3. *Def.* 5): But, as in (3. 14), it may be shewn that BC is double of BH, and FG of FK, and that the squares of EH, HB are equal to the squares EK, KF: But the square of



EH is less than the square of EK, because EH is less than EK; therefore the square of BH is greater than the square of FK, and the line BH greater than the line FK, and therefore also BC is greater than FG.

Next, let BC be greater than FG: BC shall be nearer to the centre than FG, that is, the same construction being made, EH shall be less than EK.

For, because BC is greater than FG, therefore also BH is greater than FK: But the squares of BH, HE are equal to the squares of FK, KE, of which the square of BH is greater than the square of FK, because BH is greater than FK; therefore the square of EH is less than the square of EK, and the line EH than the line EK.

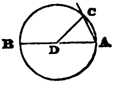
Wherefore, *The diameter &c.* Q.E.D.

PROP. XVI. THEOR.

The straight line, drawn at right angles to the diameter of a circle from the extremity of it, falls without the circle: and no straight line can be drawn from the extremity, between that straight line and the circumference, so as not to cut the circle.

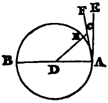
Let ABC be a circle, of which the centre is D, and the diameter AB: the straight line drawn at right angles to AB from its extremity A shall fall without the circle.

For, if not, let it fall, if possible, within the circle, as AC, and draw DC to the point C, where it meets the circumference: Then, because DA is equal to DC, the angle DAC is equal to the angle DCA: But DAC is a right angle; therefore also DCA is a right angle, and the two angles DAC, DCA are therefore equal to two right angles—which is impossible (I. 17):



Therefore the straight line drawn from A at right angles to AB does not fall within the circle: And, in like manner, it may be demonstrated that it does not fall upon the circumference: Therefore it must fall without the circle, as AE.

Also, between the straight line AE and the circumference, no straight line can be drawn from the point A, which does not cut the circle: For, if possible, let AF be between them; and from the centre D draw DG perpendicular to AF, meeting the circumference in H: Then, because AGD is a right angle, and DAG less than a right angle, therefore DA is greater than DG (1. 19): But DA is equal to DH; therefore also DH is greater than DG, the less than the greater—which is absurd: Therefore no straight line can be drawn from the point A between AE and the circumference, so as not to cut the circle.



Wherefore, *The straight line &c.* Q.E.D.

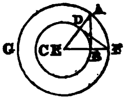
COR. From this it is manifest, that the straight line, drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; and that it touches it only in one point, because, if it met the circle in two, it would fall within it (3. 2): Also, it is evident that there can be but one straight line, which touches the circle in the same point.

PROP. XVII. PROB.

To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

First, let the given point A be without the given circle BCD: it is required to draw from A a straight line which shall touch the circle BCD.

Find E, the centre of the circle, and join AE; and from the centre E, at the distance EA, describe the circle AFG; from the point D draw DF at right angles to EA, and join EBF, AB: AB touches the circle BCD.

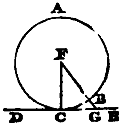


For, because E is the centre of the circles AFG, BCD, EA is equal to EF, and ED to EB: Therefore the two sides AE, EB are equal to the two FE, ED, and the angle at E is common to the two triangles AEB, FED—therefore the base AB is equal to the base FD, and the triangle AEB to the triangle FED, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle ABE is equal to the angle FDE: But FDE is a right angle, therefore also ABE is a right angle: And EB is drawn from the centre; but (3. 16. *Cor.*) a straight line, drawn at right angles to the diameter of a circle, from the extremity of it, touches the circle; therefore AB touches the circle, and it is drawn from the given point A.

But if the given point be in the circumference of the circle, as D, draw DE to the centre E, and DF at right angles to DE: DF touches the circle. Q. E. F.

PROP. XVIII. THEOR.

If a straight line touch a circle, the straight line drawn from the centre to the point of contact shall be perpendicular to the line touching the circle.



Let the straight line DE touch the circle ABC in the point C; take the centre F, and draw the straight line FC: FC is perpendicular to DE.

For, if not, from the point F draw FBG perpendicular to DE: Then, because FGC is a right angle, therefore FCG must be less than a right angle (1. 17): and to the greater angle the greater side is opposite; therefore FC is greater than FG: But FC is equal to FB; therefore also FB is greater than FG, the less than the greater—which is absurd: Therefore FG is not perpendicular to DE: And, in like manner, it may be shewn that no other line than FC can be perpendicular to DE, that is, FC is perpendicular to DE.

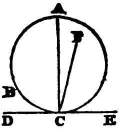
Wherefore, *If a straight line &c.* Q. E. D.

PROP. XIX. THEOR.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let the straight line DE touch the circle ABC in C, and from C let CA be drawn at right angles to DE: the centre of the circle shall be in CA.

For, if not, let, if possible, F be the centre, and join CF: Then, because DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perpendicular to DE (3. 18), and therefore FCE is a right angle: But ACE is also a right angle; therefore the angle FCE is equal to the angle ACE, the less to the greater—which is absurd: Therefore F is not the centre of the circle ABC: And, in like manner, it may be shewn that no other point, which is not in CA, can be the centre, that is, the centre is in CA.



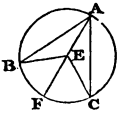
Wherefore, *If a straight line &c.* Q. E. D.

PROP. XX. THEOR.

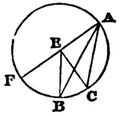
The angle at the centre of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference.

Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the same part of the circumference, BC, for their base: the angle BEC shall be double of the angle BAC.

First, let E, the centre of the circle, be within the angle BAC; join AE, and produce it to F: Then, because EA is equal to EB, the angle EAB is equal to the angle EBA; and therefore the angles EAB, EBA are double of the angle EAB: But the angle BEF (I. 32) is equal to the angles EAB, EBA; therefore also the angle BEF is double of the angle EAB: In like manner, the angle FEC is double of the angle EAC: Therefore the whole angle BEC is double of the whole angle BAC.



Next, let E, the centre of the circle, be without the angle BAC: join AE, and produce it to F: Then, as in the first case, it may be shewn that the angle FEC is double of the angle FAC, and that the angle FEB, a part of the first, is double of the angle FAB, a part of the other; therefore the remaining angle BEC is double of the remaining angle BAC.



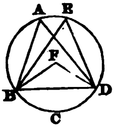
Wherefore, *The angle at the centre &c.* Q.E.D.

PROP. XXI. THEOR.

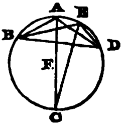
The angles in the same segment of a circle are equal to one another.

Let ABCD be a circle, and BAD, BED, angles in the same segment BAED: the angles BAD, BED shall be equal to each other.

Take F the centre of the circle ABCD: and, first, let the segment BAED be greater than a semicircle, and join BF, FD: Then, because the angle BFD at the centre and the angle BAD at the circumference have the same part of the circumference, viz. BCD, for their base; therefore (3. 20) the angle BFD is double of the angle BAD: In like manner, the angle BFD is double of the angle BED: Therefore the angle BAD is equal to the angle BED.



Next, let the segment BAED be not greater than a semicircle: draw AF to the centre, and produce it to C, and join CE: Then the segment BAEC is greater than a semicircle; and the angles in it, BAC, BEC, are equal, by the first case: For the same reason, because the segment CAED is greater than a semicircle, the angles CAD, CED are equal: Therefore the whole angle BAD is equal to the whole angle BED.



Wherefore, *The angles in the same segment &c.* Q. E. D.

PROP. XXII. THEOR.

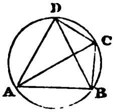
The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

Let ABCD be a quadrilateral figure inscribed in a circle: any two of its opposite angles shall be together equal to two right angles.

Join AC, BD: Then, because the three angles of every triangle are together equal to two right angles,

3

the three angles of the triangle BAC, viz. ACB, BAC, CBA, are together equal to two right angles: But the angle ADB is equal to the angle ACB (**3.** 21), because they are in the same segment ADCB; and the angle BDC is equal to the angle BAC, because they are in the same segment BADC; therefore the whole angle ADC is equal to the angles ACB, BAC: To each of these equals add the angle ABC; therefore the angles ADC, ABC are equal to the angles ACB, BAC, CBA: But ACB, BAC, CBA are together equal to two right angles; therefore also the angles ABC, ADC are together equal to two right angles: And, in like manner, it may be shewn that the angles BAD, BCD, are together equal to two right angles.

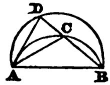


Wherefore, *The opposite angles &c.* Q.E.D.

PROP. XXIII. THEOR.

Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with one another.

If it be possible, let two similar segments of circles, viz. ACB, ADB, be upon the same side of the same straight line AB, not coinciding with one another: Now, because the circle ACB cuts the circle ADB in the two points A, B, they cannot cut one another in any other point (**3.** 10); and therefore one of the segments must fall within the other; let ACB fall within ADB, and draw the straight line BCD, and join AC, AD.



Then, because the segment ACB is similar to the segment ADB, and that similar segments of circles contain equal angles, the angle ACB is equal to the angle

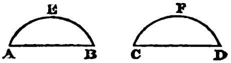
ADB, the exterior to the interior—which is impossible: therefore the similar segments ACB, ADB, upon the same side of the same straight line, AB, cannot but coincide.

Wherefore, *Upon the same &c.* Q.E.D.

PROP. XXIV. Theor.

Similar segments of circles upon equal straight lines are equal to one another.

Let AEB, CFD be similar segments of circles upon equal straight lines AB, CD: the segment AEB is equal to the segment CFD.



For if the segment AEB be applied to the segment CFD, so that the point A may be on C, and the straight line AB upon CD, the point B shall coincide with the point D, because AB is equal to CD: Therefore, the straight line AB coinciding with CD, the segment AEB must coincide with the segment CFD (3. 23), and must therefore be equal to it.

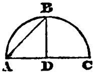
Wherefore, *Similar segments &c.* Q.E.D.

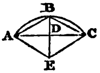
PROP. XXV. Prob.

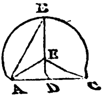
A segment of a circle being given, to describe the circle of which it is the segment.

Let ABC be the given segment of a circle: it is required to describe the circle of which it is the segment.

Bisect AC in D, and from the point D draw DB at right angles to AC, and join AB.







First, let the angles DAB, DBA be equal to one another: Then, because the angle DBA is equal to the angle DAB, therefore DB is equal to DA, and therefore also to DC: And because the three straight lines DA, DB, DC are all equal, D is the centre of the circle (3. 9): From the centre D, at the distance of any of the three DA, DB, DC, describe a circle; this shall pass through the extremities of the others, and be the circle, of which ABC is a segment: Also, because the centre D is in AC, the segment ABC is a semicircle.

Next, if the angles DAB, DBA be not equal to one another: At the point A, in the straight line BA, make the angle BAE equal to the angle ABD, and produce BD, if necessary, to E, and join EC: Then, because the angle EAB is equal to the angle EBA, the straight line EA is equal to EB: And, because AD is equal to CD, and DE common to the triangles ADE, CDE, the two sides AD, DE are equal to the two CD, DE, each to each, and the angle ADE is equal to the angle CDE, for each of them is a right angle—therefore the base AE is equal to the base CE:

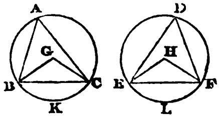
But EA was shewn to be equal to EB; therefore also EB is equal to EC, and the three straight lines EA, EB, EC are equal to one another, and therefore E is the centre of the circle: From the centre E, at the distance of any of the three EA, EB, EC, describe a circle; this shall pass through the other points, and be the circle, of which ABC is a segment: Also, it is evident, that if the angle DAB be greater than the angle DBA, the centre E falls without the segment ABC, which therefore is less than a semicircle; but if the angle DAB be less than the angle DBA, the centre E falls within the segment ABC, which is therefore greater than a semicircle: Wherefore, a segment of a circle being given, the circle is described of which it is a segment. Q.E.D.

PROP. XXVI. THEOR.

In equal circles, equal angles stand upon equal circumferences, whether they be at the centres or circumferences.

Let ABC, DEF be equal circles, and BGC, EHF equal angles at their centres, and BAC, EDF, at their circumferences, which stand on the circumferences BKC, ELF: the circumference BKC shall be equal to the circumference ELF.

Join BC, EF: Then, because the circles ABC, DEF are equal, the straight lines drawn from their centres are equal: Therefore the two sides BG, GC are equal to the two EH, HF, and the angle at G is equal to the angle at H—therefore the base BC is equal to the base EF: And, because the angle at A is equal to the angle at D, the segment BAC is similar to the segment EDF, and they are upon equal straight lines BC, EF: But *similar segments of circles upon equal straight lines are equal to one another* (3. 24); therefore the segment BAC is equal to the segment EDF: But the whole circle ABC is equal to the whole circle DEF; therefore the remaining segment BKC is equal to the remaining segment ELF, and the circumference BKC to the circumference ELF.



Wherefore, *In equal circles &c.* Q.E.D.

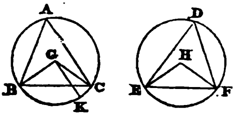
PROP. XXVII. THEOR.

In equal circles, the angles which stand upon equal circumferences are equal to one another, whether they be at the centres or circumferences.

Let ABC, DEF be equal circles, and let the angles

BGC, EHF at the centres, and BAC, EDF at the circumferences, stand upon equal circumferences BC, EF: the angle BGC shall be equal to the angle EHF, and the angle BAC to the angle EDF.

If the angle BGC be equal to the angle EHF, it is manifest (3. 20) that the angle BAC is also equal to the angle EDF: But, if not, one of them must be the greater: Let BGC be the greater, and at the point G, in the straight line BG, make the angle BGK equal to the angle EHF: Then, because equal angles at the centre stand upon equal circumferences (3. 26), therefore the circumference BK is equal to the circumference EF: But EF is equal to BC; therefore also BK is equal to BC, the less to the greater—which is absurd: Therefore the angle BGC is not unequal to the angle EHF, that is, it is equal to it: And the angle at A is half of the angle BGC, and the angle at D half of the angle EHF; therefore also the angle at A is equal to the angle at D.



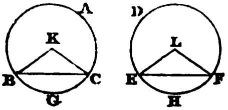
Wherefore, *In equal circles &c.* Q. E. D.

PROP. XXVIII. THEOR.

In equal circles, equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less.

Let ABC, DEF be equal circles, and BC, EF equal straight lines in them, which cut off the two greater circumferences BAC, EDF, and the two less BGC, EHF: the greater circumference BAC is equal to the greater EDF, and the less BGC to the less EHF.

Take K, L, the centres of the circles, and join BK, KC, EL, LF: Then, because the circles are equal, the straight lines from their centres are equal; therefore BK, KC are equal to EL, LF, each to each, and the base BC is equal to the base EF—therefore the angle BKC is equal to the angle ELF: But equal angles, at the centres, stand upon equal circumferences (3. 26); therefore the circumference BGC is equal to the circumference EHF: But the whole circle ABC is equal to the whole circle DEF; therefore the remaining circumference BAC is equal to the remaining circumference EDF.



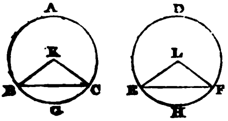
Wherefore, *In equal circles &c.* Q.E.D.

PROP. XXIX. THEOR.

In equal circles, equal circumferences are subtended by equal straight lines.

Let ABC, DEF be equal circles, and let BGC, EHF be equal circumferences, and join BC, EF: the straight line BC shall be equal to the straight line EF.

Take K, L, the centres of the circles, and join BK, KC, EL, LF: Then, because the circumference BGC is equal to the circumference EHF, the angle BKC is equal to the angle ELF (3. 27): And because the circles ABC, DEF are equal, the straight lines from their centres are equal; therefore BK, KC are equal to LF, EL, each to each, and they contain equal angles—therefore the base BC is equal to the base EF.



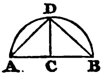
Wherefore, *In equal circles &c.* Q.E.D.

PROP. XXX. PROB.

To bisect a given circumference, that is, to divide it into two equal parts.

Let ADB be the given circumference; it is required to bisect it.

Join AB, and bisect it in C; from C draw CD at right angles to AB, and join AD, DB: the circumference ADB shall be bisected in the point D.



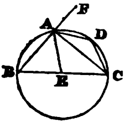
Because AC is equal to CB, and CD common to the two triangles, ACD, BCD, the two sides AC, CD are equal to the two BC, CD, each to each, and the angle ACD is equal to the angle BCD, because each of them is a right angle—therefore the base AD is equal to the base BD: But equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less (3. 28); and AD, DB are each of them less than a semicircle, because DC passes through the centre (3. 1. *Cor.*); therefore the circumference AD is equal to the circumference DB: Wherefore the given circumference is bisected in D. Q.E.F.

PROP. XXXI. THEOR.

In any circle, the angle in a semicircle is a right angle, and the angle in a segment greater than a semicircle is less than a right angle, and the angle in a segment less than a semicircle is greater than a right angle.

Let ABCD be a circle, of which the diameter is BC, and centre E; and draw CA, dividing the circle into the segments ABC, ADC, and join BA, AD, DC: the angle in the semicircle BAC is a right angle, and the angle in the segment ABC, which is greater than a semicircle, shall be less than a right angle; and the angle in the segment ADC, which is less than a semicircle, shall be greater than a right angle.

Join AE, and produce BA to F: Then, because EA is equal to EB, the angle EAB is equal to the angle EBA, and, because EA is equal to EC, the angle EAC is equal to ECA; therefore the whole angle BAC is equal to the two angles ABC, ACB: But FAC, the exterior angle of the triangle ABC, is equal to the two angles ABC, ACB (1. 32); therefore the angle BAC is equal to the angle FAC, and each of them is therefore a right angle: Therefore the angle BAC in a semicircle is a right angle.



Again, because the two angles ABC, BAC of the triangle ABC are together less than two right angles (1. 17), and that BAC is a right angle, therefore ABC must be less than a right angle, and therefore the angle in a segment ABC, greater than a semicircle, is less than a right angle.

Lastly, because ABCD is a quadrilateral figure in a circle, any two of its opposite angles are together equal to two right angles (3. 22); therefore the angles ABC, ADC are together equal to two right angles: But ABC is less than a right angle; therefore ADC is greater than a right angle, and therefore the angle in a segment ADC, less than a semicircle, is greater than a right angle.

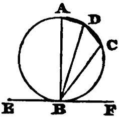
Wherefore, *The angle &c.* Q.E.D.

COR. From this it is manifest, that, if one angle of a triangle be equal to the other two, it is a right angle, because the angle adjacent to it is equal to the same two; and, when the adjacent angles are equal, each of them is a right angle.

PROP. XXXII. THEOR.

If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angle made by this line with the line touching the circle shall b equal to the angles which are in the alternate segments o the circle.

Let the straight line EF touch the circle ABCD i B, and from the point B let the straight line BD b drawn, cutting the circle: the angles, which BD make with the touching line EF, shall be equal to the angle in the alternate segments of the circle, that is, th angle FBD shall be equal to the angle in the segmen BAD, and the angle EBD to the angle in the segmen BCD.



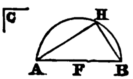
From the point B draw BA at right angles to EF and take any point C in the circumference BD, and join AD, DC, CB: Then, because the straight line EF touches the circle ABCD in the point B, and BA is drawn at right angles to the touching line from the point of contact B, the centre of the circle is in BA (3. 19): Therefore the angle ADB in a semicircle is a right angle (3. 31); and, consequently, the other two angles DAB, ABD, are together equal to a right angle: But ABF is also a right angle; therefore the angle ABF is equal to the two angles DAB, ABD: From each of these equals take away the common angle ABD; then the remaining angle FBD is equal to the angle DAB, which is in the alternate segment of the circle: Again, because ABCD is a quadrilateral figure in a circle, the opposite angles DAB, DCB are together equal to two right angles (3. 22): But the angles FBD, EBD, are together equal to two right angles; therefore the angles FBD, EBD, are equal to the angles DAB, DCB: and the angle FBD has been proved equal to the angle DAB; therefore the remaining angle EBD is equal to the angle DCB in the alternate segment of the circle.

Wherefore, *If a straight line &c.* Q. E. D.

PROP. XXXIII. PROB.

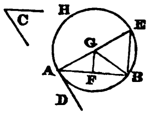
Upon a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.

Let AB be the given straight line, and C the given rectilineal angle: it is required to describe upon the given straight line AB a segment of a circle, containing an angle equal to the angle C.

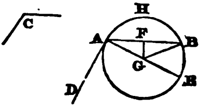


First, let the angle at C be a right angle: bisect AB in F, and from the centre F, at the distance FB, describe the semicircle AHB: then the angle AHB in a semicircle (3. 31) is equal to the right angle at C.

But if the angle C be not a right angle, at the point A, in the straight line AB, make the angle BAD equal to the angle C, and from the point A draw AE at right angles to AD; bisect AB in F, and from F draw FG at right angles to AB, and join GB.



Then, because AF is equal to FB, and FG common to the two triangles AFG, BFG, the two sides AF, FG are equal to the two BF, FG, each to each, and the angle AFG is equal to the angle BFG—therefore the base AG is equal to the base BG, and the circle described from the centre G, at the distance GA, shall pass through the point B: Let this be the circle AEB: And because from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE, therefore AD touches the circle (**3**. 16. *Cor.*): And because AB, drawn from the point of contact A, cuts the circle, the angle DAB is equal to the angle in the alternate segment AHB (**3**. 32): But the angle DAB is equal to the angle C; therefore also the angle in the segment AHB is equal to the angle C: Wherefore, upon the given straight line AB, the segment AHB of a circle is described, which contains an angle equal to the given angle at C. Q.E.F.



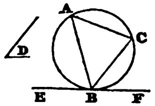
PROP. XXXIV. PROB.

From a given circle to cut off a segment, which shall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle: it is required to cut off from the circle ABC a segment that shall contain an angle equal to the given angle D.

Draw (**3**. 17) the straight line EF touching the circle ABC in the point B; and at the point B, in the straight line BF, make the angle FBC equal to the angle D.

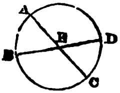
Then, because the straight line EF touches the circle ABC, and BC is drawn from the point of contact B, the angle FBC is equal to the angle in the alternate segment, BAC, of the circle (**3**. 32): But the angle FBC is equal to the angle D; therefore the angle in the segment BAC is equal to the angle D: Wherefore from the given circle ABC has been cut off the segment BAC, containing an angle equal to the given angle D. Q.E.F.



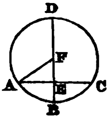
PROP. XXXV. THEOR.

If two straight lines within a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

Let the two straight lines AC, BD, within the circle ABCD, cut one another in the point E: the rectangle contained by AE, EC shall be equal to the rectangle contained by BE, ED.

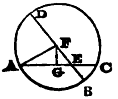


If AC, BD pass, each of them, through the centre, so that E is the centre, it is evident that AE, EC, BE, ED, being all equal, the rectangle AE, EC is equal to the rectangle BE, ED.

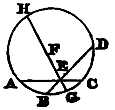


But let one of them, BD, pass through the centre, and cut the other AC, which does not pass through the centre, at right angles, in the point E: Bisect BD in F, which must be the centre of the circle ABCD, and join AF: Then, because BD, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles in E, AC is bisected in E (**3**. 3): And because the straight line BD is divided into two equal parts in the point F, and into two unequal in the point E, the rectangle BE, ED, together with the square of EF, is equal to the square of FB (**2**. 5), that is, to the square of AF: But the squares of AE, EF are equal to the square of AF; therefore the rectangle BE, ED, together with the square of EF, is equal to the squares of AE, EF: Take away the common square of EF, and the remainder, the rectangle BE, ED, is equal to the remainder, the square of AE, that is, to the rectangle AE, EC.

Next, let BD, which passes through the centre, cut the other AC, which does not pass through the centre, in E, but not at right angles: bisect, as before, BD in F, the centre of the circle; join AF, and from F draw FG perpendicular to AC: Then AC is bisected in G; therefore the rectangle AE, EC, together with the square of EG, is equal to the square of AG: To each of these equals add the square of GF; therefore the rectangle AE, EC, together with the squares of EG, GF, is equal to the squares of AG, GF: But the squares of EG, GF are equal to the square of EF, and the squares of AG, GF are equal to the square of AF; therefore the rectangle AE, EC, together with the square of EF, is equal to the square of AF, that is, to the square of FB: But the square of FB is equal to the rectangle BE, ED, together with the square of EF; therefore the rectangle AE, EC, together with the square of EF, is equal to the rectangle BE, ED, together with the square of EF: Take away the common square of EF, and the remainder, the rectangle AE, EC, is equal to the remainder, the rectangle BE, ED.



Lastly, let neither of the straight lines AC, BD pass through the centre: take the centre F, and through E, the intersection of the straight lines AC, BD, draw the diameter GEFH: Then, because, as has been shewn, the rectangle GE, EH is equal to the rectangle AE, EC, and also to the rectangle BE, ED, therefore the rectangle AE, EC is equal to the rectangle BE, ED.



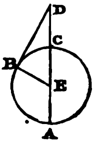
Wherefore, *If two straight lines &c.* Q.E.D.

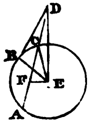
PROP. XXXVI. THEOR.

If from any point without a circle two straight lines be drawn, one of which cuts the circle and the other touches it, the rectangle, contained by the whole line which cuts the circle and the part of it without the circle, shall be equal to the square of the line which touches it.

Let D be any point without the circle ABC, and DCA, DB two straight lines drawn from it, of which DCA cuts the circle, and DB touches it: the rectangle AD, DC shall be equal to the square of DB.

First, let DCA pass through the centre E, and join EB: Then EBD is a right angle (3. 18): And because AC is bisected in E, and produced to D, the rectangle AD, DC, together with the square of EC, is equal to the square of ED (2. 6): But EC is equal to EB; therefore the rectangle AD, DC, together with the square of EB, is equal to the square of ED: But the square of ED is equal to the squares of EB, BD, because EBD is a right angle; therefore the rectangle AD, DC, together with the square of EB, is equal to the squares of EB, BD: Take away the common square of EB; therefore the remainder, the rectangle AD, DC, is equal to the square of DB.



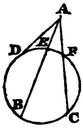


Next, let DCA not pass through the centre of the circle ABC; take the centre E, and draw EF perpendicular to AC, and join EB, EC, ED: Then because EF, which passes through the centre, cuts AC, which does not pass through the centre, at right angles, it also bisects it; therefore AF is equal to FC: And because AC is bisected in F, and produced to D, the rectangle AD, DC, together with the square of

FC, is equal to the square of FD: To each of these equals add the square of FE; therefore the rectangle AD, DC, together with the squares of CF, FE, is equal to the squares of DF, FE: But the square of CE is equal to the squares of CF, FE, because CFE is a right angle, and the square of DE is equal to the squares of DF, FE; therefore the rectangle AD, DC, together with the square of CE, is equal to the square of DE: And CE is equal to BE; therefore the rectangle AD, DC, together with the square of BE, is equal to the square of DE: But the squares of DB, BE are equal to the square of DE, because DBE is a right angle; therefore the rectangle AD, DC, together with the square of BE, is equal to the squares of DB, BE: Take away the common square of BE; therefore the remainder, the rectangle AD, DC, is equal to the square of DB.

Wherefore, *If from any point &c.* Q. E. D.

COR. If from any point without a circle, there be drawn two straight lines cutting it, as AB, AC, the rectangles, contained by the whole lines and the parts of them without the circle, are equal to one another, viz. the rectangle BA, AE, to the rectangle CA, AF; for each of them is equal to the square of the straight line AD, which touches the circle.

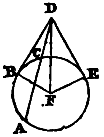


PROP. XXXVII. THEOR.

If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it, then, if the rectangle, contained by the whole line which cuts the circle and the part of it without the circle, be equal to the square of the line which meets it, the line which meets shall touch the circle.

Let any point D be taken without the circle ABC, and from it let two straight lines, DCA, DB, be drawn, of which DCA cuts the circle, and DB meets it, and let the rectangle AD, DC be equal to the square of DB: DB shall touch the circle.

Draw the straight line DE, touching the circle ABC (3. 17); find its centre F, and join FB, FD, FE: Then FED is a right angle (3. 18): And because DE touches the circle ABC, and DCA cuts it, the rectangle AD, DC is equal to the square of DE (3. 36): But (*hyp.*) the rectangle AD, DC is equal to the square of DB; therefore the square of DE is equal to the square of DB, and the straight line DE to the straight line DB: And EF is equal to BF; therefore the two DE, EF are equal to the two DB, BF, each to each, and the base DF is common to the two triangles DEF, DBF—therefore the angle DEF is equal to the angle DBF: But DEF is a right angle; therefore also DBF is a right angle: And BF, if produced, is a diameter; and the straight line which is drawn at right angles to a diameter, from the extremity of it, touches the circle (3. 16); therefore DB touches the circle ABC.



Wherefore, *If from a point &c.* Q. E. D.

PROBLEMS.

BOOK III.

1. DESCRIBE a circle of given radius, which shall pass through two given points.

2. If with the vertex of an isosceles triangle, as centre, a circle be described cutting the base or base produced, the parts of it intercepted between the circle and the extremities of the base will be equal.

3. If two circles cut each other, any two parallel lines drawn through the points of section to cut the circles are equal.

4. If two circles cut each other, draw through one of the points of section a line which shall be terminated in the circumferences and be bisected in that point.

5. A chord PAQ cuts the diameter of a circle in A in an angle which is half a right angle: shew that the squares of AP and AQ are together double of the square of the radius.

6. If two chords intersect in a circle, the difference of their squares is equal to the difference of the squares of the difference of the segments.

7. Two parallel chords in a circle are respectively six and eight inches in length, and are one inch apart: how many inches in length is the diameter?

8. Draw a line cutting two concentric circles, so that the part of it intercepted by the circumference of the greater may be double the part intercepted by that of the less.

9. If two circles cut each other, the greatest line that can be drawn through the point of intersection is that which is parallel to the line joining their centres.

10. Describe three equal circles touching one another, and also another which shall touch them all three.

11. How many equal circles can be described around another circle of the same magnitude, touching it and one another?

12. Describe a circle which shall pass through a given point, and touch a given circle in a given point, the two points not being in a tangent to the given circle.

13. Describe a circle which shall touch a given circle in a given point, and also touch a given straight line.

14. If from any point without a circle two lines be drawn making equal angles with the line through the centre from that point, they will cut off equal segments from the circle.

15. Through a given point within a circle draw the least possible chord.

16. Of all lines which touch the interior and are bounded by the exterior of two circles which touch internally, the greatest is that which is parallel to the common tangent.

17. Shew that the two tangents to a circle drawn from the same point without it are equal to one another; and hence prove that the sums of the opposite sides of any quadrilateral described about a circle are equal, and the angles subtended at the centre of the circle by any two opposite sides together equal to two right angles.

18. If any line be drawn touching a circle, the part of it intercepted between the tangents at the extremities of any diameter subtends at the centre a right angle.

19. In the diameter of a circle produced determine a point from which a tangent drawn to the circle shall be equal to the diameter.

20. Describe a circle which shall pass through a given point, have a given radius, and touch a given line.

21. Describe a circle whose centre shall be in the perpendicular of a given right-angled triangle, and which shall pass through the right angle and touch the hypothenuse.

22. A is any point in the diameter (or diameter produced) of a circle, whose centre is O, OB a radius perpendicular to the diameter: if AB cut the circle in P, and the tangent at P cut AO in C, shew that $AC = CP$.

23. A common tangent is drawn to two circles which touch externally: if a circle be described on that part of it which lies between the points of contact, as diameter, it will pass through the point of contact of the two circles, and be touched by the line joining their centres.

24. Describe a circle with given radius and its centre in a given line, which shall touch another given line.

25. Describe a circle, which shall touch a given line in a given point, and also touch a given circle.

26. Draw a common tangent to two circles, when the points of contact are (i) on the same side, and (ii) on opposite sides, of the line joining their centres.

27. Draw a line which shall touch a given circle, and make with a given line a given angle.

28. Describe two circles of given radii, which shall touch each other, and the same given line on the same side of it.

29. If two circles touch each other, and parallel diameters be drawn, then lines which join the extremities of these diameters will pass through the point of contact.

30. The line, drawn from the vertex of an equilateral triangle to meet the circumscribing circle in any point, is equal to the *sum* or *difference* of the two lines drawn from the extremities of the base to that point, according as it *does* or *does not* cut the base.

31. AB, AC are any two chords of a circle, D, E, the bisections of the arcs AB, AC: let DE cut AB, AC in F, G, and shew that AF = AG.

32. If the opposite angles of a quadrilateral be together equal to two right angles, shew that a circle may be described about it, and find its centre and radius.

33. ABCD is a parallelogram; draw CE perpendicular to the diagonal BD, and shew that perpendiculars upon AB, AD at the points B, D, will intersect in CE.

34. The circles described on the three sides of a triangle, so as to pass through the points of intersection of the perpendiculars upon them from the opposite angles, are equal to each other.

35. Two circles intersect in A, B, the centre of one being in the circumference of the other: draw any chord ACD cutting them both, and shew that CB = CD.

36. If from any two points in the circumference of a circle there be drawn two lines to a point in any tangent to the circle, they will make the greatest angle when drawn to the point of contact.

37. Given three points in a circle: shew how we may find any number of other points, without knowing the position of the centre.

38. If through the angles of a quadrilateral, lines bisecting them be drawn, the points in which each line intersects the adjacent ones will all lie in the circumference of a circle.

39. ABC is a semicircle, ADC a quadrant, upon the same line AC and on the same side of it; from any point B in the semicircle draw BA, BDC, and shew that BA and BD are equal, and that the longer only of the lines AB, AC, can cut the circle ADC.

40. If any chord of a circle be bisected by another and produced to meet the tangents at the extremities of the bisecting line, the parts intercepted between the tangents and the circumferences are equal.

41. From the extremities A, C of a given circular arc, equal arcs AB, CD are measured in opposite directions: shew that the chords AC, BD are parallel.

42. The arcs intercepted between any two parallel chords of a circle are equal; and if any two chords of a circle intersect each other, the sum of the arcs intercepted by them is equal to the sum of the arcs intercepted by diameters parallel to them.

43. A, B, C, A′, B′, C′ are points in the circumference of a circle: if AB, AC be respectively parallel to A′B′, A′C′, shew that BC′ is parallel to B′C.

44. If two equal circles cut each other, and from either point of intersection a circle be described cutting them, the point where this circle cuts them and the other point of intersection of the equal circles are in the same straight line.

45. If two equal circles cut each other and from either point of intersection a line be drawn cutting the circumferences, the part of it between them will be bisected by the circle, whose diameter is the common chord of the equal circles.

46. If two circles cut each other, and any two points be taken in the circumference of one *through* which lines are drawn from the points of intersection cutting that of the other, the lines, joining the points of section with the latter circle of those drawn *through* the same point, will be equal.

47. A, B are given points: if AC, BC are drawn, making a given angle with each other, shew that the line bisecting that angle passes through a fixed point.

48. If perpendiculars be dropped from the extremities of any diameter upon any chord of a circle, the parts of the chord, intercepted between them and the circle, will be equal, and the less perpendicular shall be equal to the segment of the greater contained between the chord and circumference.

49. If the base of a triangle be bisected by the diameter of the circumscribing circle, and from the extremity of that diameter a perpendicular be let fall on the longer side, it will divide that side into segments, one of which is equal to the semisum and the other to the semidifference of the sides.

50. Given the radius of a circle, that touches two given lines not parallel, determine its centre.

51. Find a point in the diameter produced of a given circle, such that, if tangents be drawn from it to the circle, the concave part of the circumference may be double of the convex.

52. The line drawn through the bisection of any arc of a circle, parallel to its chord, is a tangent to the circle at that point: and the radius which bisects the chord of an arc bisects also the arc.

53. If, from each extremity of two adjacent arcs of a circle, lines be drawn through two given points in the opposite circumference and produced till they meet, the angles formed by these lines will be equal.

54. If from the middle point of any arc of a circle, a perpendicular be drawn to the diameter through one extremity, it will bisect the segment of the chord cut off by the line, which joins the middle point aforesaid and the other extremity of the diameter.

55. Two equal circles pass through each others' centres A, B: if any common chord CEFD, be drawn parallel to AB, shew that the figures ACEB, AFDB, are parallelograms; and, if AF be produced to meet the circumferences again in G and H, shew that EF and FH, as also CD and GH, are equal.

56. ABC is a triangle inscribed in a circle, DEF a diameter cutting BC at right angles in E: shew that the difference of the angles at B and C is double of the angle AFD.

57. ACB, ADB are arcs of equal circles on the same line AB and on the same side of it: draw any

chord ACD cutting them both, and shew that BC and BD are equal.

58. If two chords in a circle intersect each other, the angle between them is half the angle at the centre subtended by an arc, which is equal to the *sum* or *difference* of the arcs intercepted by them, according as they intersect *within* or *without* the circle. Shew also that if they intersect at right angles, the sum of any two opposite arcs equals half the circumference.

59. If circles be described on the two sides of a right-angled triangle as diameters, they will be touched by a circle, whose centre is the bisection of the hypothenuse, and diameter equal to the sum of the sides.

60. If perpendiculars A*a*, B*b*, C*c* be drawn from the angular points of a triangle ABC upon the sides, shew that they will bisect the angles of the triangle *abc*.

61. If a circle be described on the radius of another circle, any line drawn from the point where they meet to the outer circumference is bisected by the inner.

62. The circles described on the sides of any triangle as diameters will intersect in the sides, or sides produced, of the triangle.

63. If from the extremity of the diameter of a circle any chord be drawn, shew that its middle point will always lie in the circumference of a given circle.

64. If an equilateral triangle be inscribed in a circle, and the adjacent arcs cut off by two of its sides be bisected, the line joining the points of section will be trisected by the sides.

65. The vertical angle of any oblique-angled triangle inscribed in a circle is greater or less than a right-angle by the angle contained by the base and the diameter drawn from the extremity of the base: and no parallelogram can be inscribed in a circle except a rectangle.

66. From one extremity of a line, which cannot be produced, draw a line perpendicular to it.

67. ABC is an equilateral triangle, D, E, F the bisections of its sides; shew that DF touches the circle CDE.

68. If upon any radius of a circle a circle be described, and any other two radii be drawn cutting it, the chord of the arc intercepted by them is equal to the perpendicular from the extremity of one upon another.

69. If two circles touch each other and any two lines be drawn passing through the point of contact, the chords of the intercepted arcs will be parallel.

70. If a semicircle be described on the side of a quadrant, and from any point in the quadrantal arc, a radius be drawn, the part of it between the quadrant and the semicircle is equal to the perpendicular from the same point on the common tangent.

71. Given one angle, the side opposite, and the sum of the other two sides, construct the triangle.

72. Given the area and hypothenuse of a right-angled triangle: construct it.

73. Given the base, vertical angle, and altitude: construct the triangle.

74. Of all triangles on the same base and having the same vertical angle, the isosceles is greatest; and, conversely, of all triangles on the same base and between the same parallels, the isosceles has the greatest vertical angle.

75. Through three given points draw three lines, so as to make an equilateral triangle.

76. If two circles cut each other, draw through the point of section a line cutting both the circles and equal to a given line; and hence through three given points draw lines, so as to make a triangle equal in all respects to a given triangle.

77. Find a point within a triangle at which the three sides shall subtend equal angles: and shew that

if on the three sides of a triangle towards the same parts similar segments of circles be described, and from the extremities of the base tangents to the segment upon it be drawn to cut the other circumferences, the points of intersection and the vertex will be in the same line parallel to the base.

78. AB is the diameter of a circle, CD a chord perpendicular to AB; if through any point P in CD a chord APQ be drawn, the rectangle AP, AQ is constant.

79. Given the area, one angle, and a line drawn from one of the others bisecting the opposite side: construct the triangle.

80. Describe a circle which shall pass through two given points and touch a given line: and shew that, of all triangles upon a given base and between the same parallels, the isosceles has the greatest vertical angle.

81. Describe a circle which shall pass through two given points and touch a given line in a given point.

82. Through a given point describe a circle touching two given lines.

83. Describe a circle touching two given lines and a given circle.

84. Describe an isosceles triangle, having given the base angle and the perpendicular from it upon the opposite side.

85. If from the centre of a circle a line be drawn to any point in the chord of an arc, the square of that line together with the rectangle of the segments of the chord will be equal to the square of the radius.

86. Let the diameter of a circle cut at right angles in A a given chord, and let any other chord, BC, cut the same chord in D: then the sum of the square of AD and the rectangle of BD, CD is constant.

87. If through any point within or without a circle two lines cut each other at right angles, the sums of the squares of the two lines joining their extremities, and

also the sum of the squares of the four segments, are each equal to the square of the diameter.

88. If there be any three circles in a plane, and, through the centres of each two of them, a circle be described touching the third, the lines, joining the centre of each of the circles with the point in which the circles passing through it intersect, will meet in the same point.

89. Two points are taken in the diameter of a circle at any equal distances from the centre: through one of these draw any chord, and join its extremities with the other point; and shew that the triangle thus formed has the sum of the squares of its sides invariable.

90. ABC is a triangle whose acute vertex is A; shew that the square of BC is less than the squares of AB, AC by twice the square of the line drawn from A to touch the circle on BC as diameter.

91. If any number of circles be drawn through two given points cutting a given circle, the lines which join the points of intersection shall all meet the line joining the two given points in the same point.

92. If three circles cut each other two and two, their three chords of intersection meet in a point.

93. Let ACDB be a semicircle, whose diameter is AB, and AD, BC any two chords intersecting in P; shew that $AB = AD.AP + BC.BP$.

94. Let a common tangent be drawn to any number of circles which touch each other internally, and with any point of this tangent as centre describe a circle cutting the other circles; then if from this centre lines be drawn through the intersections of the circles, the segments of these lines within each circle will be equal.

95. A rod of given length is moved betwixt two fixed straight lines CP, CQ; the perpendiculars from P, Q upon CP, CQ meet in R, and those from P, Q upon CQ, CP meet in S; shew that the loci of R, S are circles with common centre C.

96. If a semicircle be inscribed in a right-angled triangle so as to touch the hypothenuse and perpendicular, and from the end of its diameter a line be drawn through the point of contact to meet the perpendicular produced, the part produced will be equal to the perpendicular.

97. The circle described through any two of the angular points of a triangle and the intersection of the perpendiculars from the angles on the opposite sides will be equal to the circumscribing circle of the triangle.

98. AB, CD are chords of a circle, centre O, intersecting at right angles in E: shew that the squares of AB, CD together with four times the square of OE are double of the square of the diameter.

99. If ABCD be a parallelogram, and if a circle be described through A, cutting the sides AB, AD, and the diagonal AC, in F, H, G, respectively, then

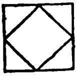
$$AB.AF + AD.AH = AC.AG.$$

100. Given the vertical angle, the difference of the sides containing it, and the difference of the segments of the base made by a perpendicular from the vertex: construct the triangle.

BOOK IV.

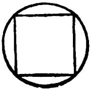
DEFINITIONS.

I. A RECTILINEAL figure is said to be inscribed in another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.

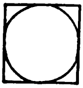


II. In like manner, a figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

III. A rectilineal figure is said to be inscribed in a circle, when all the angles of the inscribed figure are upon the circumference of the circle.

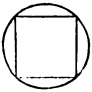


IV. A rectilineal figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.



V. In like manner, a circle is said to be inscribed in a rectilineal figure, when the circumference of the circle touches each side of the figure.

VI. A circle is said to be described about a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.

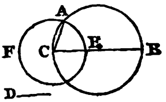


VII. A straight line is said to be placed in a circle, when the extremities of it are in the circumference of the circle.

PROP. I. PROB.

In a given circle to place a straight line, equal to a given straight line which is not greater than the diameter of the circle.

Let ABC be the given circle, and D the given straight line, not greater than the diameter of the circle: it is required to place in the circle ABC a straight line equal to D.



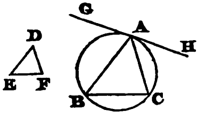
Draw BC the diameter of the circle ABC: Then, if BC is equal to D, the thing required is done; for in the circle ABC, a straight line BC is placed equal to D: But, if it is not, BC is greater than D: make CE equal to D, and from the centre C, at the distance CE, describe the circle AEF, and join CA: Then, because C is the centre of the circle AEF, CA is equal to CE: But CE is equal to D; therefore CA is equal to D: Wherefore, in the circle ABC, a straight line is placed equal to the given straight line D, which is not greater than the diameter of the circle. Q.E.F.

PROP. II. PROB.

In a given circle to inscribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle: it is required to inscribe in the circle ABC a triangle equiangular to DEF.

Draw the straight line GAH touching the circle in the point A (3. 17); and, at the point A, in the straight line AH, make the angle HAC equal to the angle DEF, and at the point A, in the straight line AG, make the angle GAB equal to the angle DFE, and join BC: Then, because GAH touches the circle ABC, and AC is drawn from the point of contact cutting the circle, the angle HAC is equal to the angle ABC in the alternate segment of the circle (3. 32); but the angle HAC is equal to the angle DEF; therefore also the angle ABC is equal to the angle DEF: For the like reason, the angle ACB is equal to the angle DFE: Therefore the remaining angle BAC is equal to the remaining angle EDF: Wherefore the triangle ABC is equiangular to the given triangle DEF, and it is inscribed in the given circle ABC. Q. E. F.

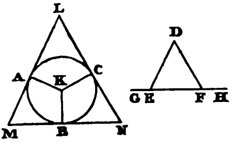


PROP. III. PROB.

About a given circle to describe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle: it is required to describe about the circle ABC a triangle equiangular to the triangle DEF.

Produce EF both ways to the points G, H; find the centre K of the circle ABC, and from it draw any straight line KB; at the point K in the straight line BK, make the angle BKA equal to the angle DEG, and the angle BKC equal to the angle



DFH; and through the points A, B, C, draw the straight lines LAM, MBN, NCL, touching the circle ABC (3. 17).

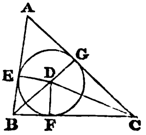
Then, because LM, MN, NL touch the circle ABC in the points A, B, C, to which from the centre are drawn KA, KB, KC, therefore the angles at the points A, B, C are right angles (3. 18): And because the four angles of the quadrilateral figure AKBM are equal to four right angles (for it can be divided into two triangles), and that two of them KAM, KBM are right angles, therefore the other two AKB, AMB are together equal to two right angles: But the angles DEG, DEF are also together equal to two right angles; therefore the angles AKB, AMB are equal to the angles DEG, DEF, of which AKB is equal to DEG, and therefore the remaining angle AMB, or LMN, is equal to the remaining angle DEF: And in like manner, the angle LNM may be shewn to be equal to DFE: And therefore the remaining angle MLN is equal to the remaining angle EDF: Wherefore the triangle LMN is equiangular to the given triangle DEF, and it is described about the given circle ABC. Q. E. F.

PROP. IV. PROB.

To inscribe a circle in a given triangle.

Let ABC be the given triangle: it is required to inscribe a circle in the triangle ABC.

Bisect the angles ABC, ACB by the straight lines BD, CD, meeting one another in the point D, from which draw DE, DF, DG perpendiculars to AB, BC, CA.



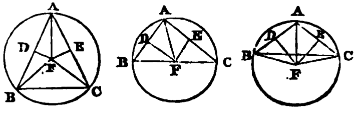
Then, because the angle DBE is equal to the angle DBF (for the angle ABC is bisected by BD), and that the right angle DEB is equal to the right angle DFB, the two triangles DEB, DFB have two angles of the one equal to two angles of the other, each to each, and the side DB, which is opposite to one of the equal angles in each, common to both—therefore their other sides are equal, and therefore DE is equal to DF: For the like reason, DG is equal to DF: Therefore the three straight lines DE, DF, DG are equal to one another, and the circle described from the centre D, at the distance of any one of them, shall pass through the extremities of the other two, and shall touch the straight lines AB, BC, CA, because the angles at the points E, F, G are right angles, and *the straight line which is drawn from the extremity of a diameter at right angles to it, touches the circle* (3. 16): Wherefore the straight lines AB, BC, CA do each of them touch the circle, and the circle EFG is inscribed in the given triangle ABC. Q.E.F.

PROP. V. PROB.

To describe a circle about a given triangle.

Let ABC be the given triangle: it is required to describe a circle about the triangle ABC.

Bisect AB, AC, in the points D, E, and from D, E, draw DF, EF at right angles to AB, AC; DF, EF produced, will meet one another; (for, if they do not meet, they are parallel, and therefore AB, AC, which are at right angles to them, are parallel—which is



absurd;) let them meet in F, and join FA; also, if the point F be not in BC, join BF, CF.

Then, because AD is equal to BD, and DF common, and at right angles to AB, therefore the base FA is equal to the base FB: In like manner, it may be shewn that FA is equal to FC; and therefore FB is equal to FC, and FA, FB, FC are equal to one another: Wherefore the circle described from the centre F, at the distance of any one of them, shall pass through the extremities of the other two, and be described about the given triangle ABC. Q.E.F.

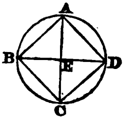
COR. And it is manifest, that if the centre of the circle fall within the triangle, each of its angles is less than a right angle, each of them being in a segment greater than a semicircle; but, if the centre be in one of the sides of the triangle, the angle opposite to this side, being in a semicircle, is a right angle; and if the centre fall without the triangle, the angle opposite to the side beyond which it falls, being in a segment less than a semicircle, is greater than a right angle: Wherefore, if the given triangle be acute-angled, the centre of the circle will fall within it; if it be right-angled, the centre will be in the side opposite to the right angle; and if it be obtuse-angled, the centre will fall without the triangle, beyond the side opposite to the obtuse angle.

PROP. VI. PROB.

To inscribe a square in a given circle.

Let ABCD be the given circle: it is required to inscribe a square in ABCD.

Draw two diameters AC, BD, intersecting one another at right angles in E, and join AB, BC, CD, DA: Then, because BE is equal to ED (for E is the centre), and that EA is common, and at right angles to BD—therefore the base AB is equal to the base AD: And, for the like reason, BC, CD are each of them equal to BA, or AD: And therefore the quadrilateral figure ABCD is equilateral: It is also rectangular; for the straight line BD being the diameter of the circle ABCD, BAD is a semicircle, and therefore the angle BAD is a right angle (3. 31): For the like reason, each of the angles ABC, BCD, CDA is a right angle: And therefore the quadrilateral figure ABCD is rectangular: And it has been shewn to be equilateral; therefore it is a square, and it is inscribed in the given circle ABCD. Q.E.F.

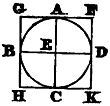


PROP. VII. PROB.

To describe a square about a given circle.

Let ABCD be the given circle: it is required to describe a square about it.

Draw two diameters AC, BD, at right angles to one another, and through the points A, B, C, D, draw FG, GH, HK, KF touching the circle (3. 17): Then, because FG touches the circle ABCD, and EA is drawn from the centre E to the point of contact A, the angles at A are right angles (3. 18): For the like reason, the angles at the points B, C, D are right angles: And because the angle AEB is a right angle, as also the angle EBG, therefore GH is parallel to AC (1. 28): For the like reason, AC is parallel to FK: And, in like manner, GF, HK may be shewn to be, each of them, parallel to BED: Therefore the figures GK, AH, AK, BF, BK,



are parallelograms; and GF is therefore equal to HK, and GH to FK: And because AC is equal to BD, and that AC is equal to each of the two GH, FK, and BD to each of the two GF, HK, therefore GH, FK are each of them equal to GF, or HK; and therefore the quadrilateral figure FGHK is equilateral: It is also rectangular; for, AEBG being a parallelogram, and AEB a right angle, therefore also AGB is a right angle (I. 34): And, in like manner, it may be shewn that the angles at F, H, K are right angles; therefore the quadrilateral figure FGHK is rectangular: And it was shewn to be equilateral; therefore it is a square, and it is described about the given circle ABCD. Q.E.F.

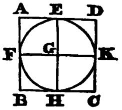
PROP. VIII. PROB.

To inscribe a circle in a given square.

Let ABCD be the given square: it is required to inscribe a circle in ABCD.

Bisect each of the sides AB, AD in the points F, E, and through E draw EH parallel to AB or DC, and through F draw FK parallel to AD or BC.

Then each of the figures AK, KB, AH, HD, AG, GC, BG, GD is a parallelogram; and their opposite sides are equal: And because AD is equal to AB, and that AE is the half of AD, and AF the half of AB, therefore AE is equal to AF, and therefore also the sides opposite to these are equal, viz. FG to GE: In like manner it may be shewn that GH, GK are each of them equal to GF or GE: Therefore the four straight lines GE, GF, GH, GK are equal to one another, and the circle described from the centre G, at the distance of any one of them, shall pass through

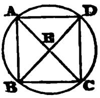


the extremities of the other three, and touch the straight lines AB, BC, CD, DA, because the angles at the points, E, F, H, K, are right angles (3. 16): Therefore each of the straight lines AB, BC, CD, DA touches the circle, which is therefore inscribed in the square ABCD. Q.E.F.

PROP. IX. Prob.

To describe a circle about a given square.

Let ABCD be the given square: it is required to describe a circle about it.

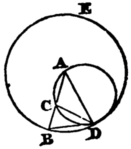


Join AC, BD, cutting one another in E: Then, because AB is equal to AD, and AC common to the two triangles BAC, DAC, the two sides BA, AC are equal to the two DA, AC, each to each, and the base BC is equal to the base DC—therefore the angle BAC is equal to the angle DAC, and the angle BAD is bisected by the straight line AC: In like manner, it may be shewn that the angles ABC, BCD, CDA are severally bisected by the straight lines BD, AC: Therefore, because the angle DAB is equal to the angle ABC, and that the angle EAB is the half of DAB, and EBA the half of ABC, therefore the angle EAB is equal to the angle EBA, and the side EA to the side EB: And, in like manner, it may be shewn that the straight lines EC, ED are each of them equal to EA, or EB: Therefore the four straight lines EA, EB, EC, ED are equal to one another, and the circle described from the centre E, at the distance of any one of them, shall pass through the extremities of the other three, and be described about the given square ABCD. Q.E.F.

PROP. X. PROB.

To describe an isosceles triangle, having each of the angles at the base double of the third angle.

Take any straight line AB, and divide it in the point C, so that the rectangle AB, BC may be equal to the square of AC (**2**. 11); from the centre A, at the distance AB, describe the circle BDE, in which place the straight line BD equal to AC (**4**. 1), which is not greater than the diameter of the circle BDE, and join DA: the triangle ABD is such as is required, that is, each of the angles ABD, ADB is double of the third angle BAD.



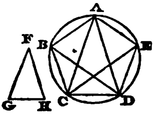
Join CD, and about the triangle ACD describe a circle: Then, because the rectangle AB, BC is equal to the square of AC, and that AC is equal to BD, therefore the rectangle AB, BC is equal to the square of BD: And because from the point B, without the circle ACD, two straight lines BCA, BD are drawn to the circumference, one of which cuts, and the other meets the circle, and that the rectangle AB, BC, contained by the whole of the cutting line, and the part of it without the circle, is equal to the square of BD which meets it, therefore BD touches the circle ACD (**3**. 37): And because BD touches the circle ACD, and that DC is drawn from the point of contact D cutting the circle, the angle BDC is equal to the angle CAD in the alternate segment of the circle (**3**. 32): To each of these add the angle CDA; therefore the whole angle BDA is equal to the two angles CAD, CDA: But the exterior angle BCD is equal to the angles CAD, CDA; therefore also the angle BDA is equal to the angle BCD: But the angle BDA is equal to the angle DBA, because AD is equal to AB; therefore each of the angles BDA, DBA, is equal to the angle BCD: Now, because the angle DBA, or DBC, is equal to the angle BCD, the side DB is equal to the side DC: But DB was made equal to CA, therefore also CA is equal to CD, and the angle CAD to the angle CDA; and therefore the angles CAD, CDA are together double of the angle CAD: But the angle BCD is equal to the angles CAD, CDA; therefore also the angle BCD is double of the angle CAD: And the angle BCD is equal to each of the angles BDA, DBA; therefore each of the angles BDA, DBA is double of the angle BAD: Wherefore an isosceles triangle ABD has been described, having each of the angles at the base double of the third angle. Q.E.F.

PROP. XI. Prob.

To inscribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given circle: it is required to inscribe an equilateral and equiangular pentagon in the circle ABCDE.

Describe an isosceles triangle FGH, having each of the angles at G, H, double of the angle at F (4. 10); and in the circle ABCDE, inscribe the triangle ACD, equiangular to the triangle FGH (4. 2), so that the angle CAD may be equal to the angle at F, and each of the angles ACD, ADC equal to the angle at G or H, and therefore each double of the angle CAD; bisect the angles ACD, ADC by the straight lines CE, DB, and join AB, BC, AE, ED: ABCDE is the pentagon required.



For, because each of the angles ACD, ADC is double of CAD, and that they are bisected by the straight lines CE, DB, therefore the five angles ADB, BDC, CAD, DCE, ECA, are equal to one another: But equal angles stand upon equal circumferences (3. 26); therefore the five circumferences, AB, BC, CD, DE, EA, are equal to one another: And equal circumferences are subtended by equal straight lines (3. 29); therefore the five straight lines AB, BC, CD, DE, EA are equal to one another; and therefore the pentagon ABCDE is equilateral: It is also equiangular: For, because the circumference AB is equal to the circumference DE, add to each the circumference BCD; therefore the whole circumference ABCD is equal to the whole circumference BCDE: And the angle AED stands on the circumference ABCD, and the angle BAE on the circumference BCDE; therefore the angle AED is equal to the angle BAE (3. 27): For the like reason, each of the angles ABC, BCD, CDE is equal to the angle AED or BAE; therefore the pentagon ABCDE is equiangular; and it has been shewn to be equilateral: Wherefore, in the given circle, an equilateral and equiangular pentagon has been inscribed. Q.E.F.

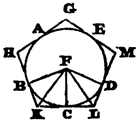
PROP. XII. PROB.

To describe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be the given circle: it is required to describe an equilateral and equiangular pentagon about the circle ABCDE.

Let the angles of a pentagon, inscribed in the circle by the last proposition, be in the points A, B, C, D, E, so that the circumferences AB, BC, CD, DE, EA are equal; and through the points A, B, C, D, E, draw GH, HK, KL, LM, MG, touching the circle; take the centre F, and join FB, FK, FC, FL, FD.

Then, because the straight line KL touches the circle ABCDE in the point C, to which FC is drawn from the centre, FC is perpendicular to KL (3. 18), and therefore each of the angles at C is a right angle: For the like reason, the angles at the points B, D are right angles: And because FCK is a right angle, the square of FK is equal to the squares of FC, CK, and, for the like reason, the square of FK is equal to the squares of FB, BK; therefore the squares of FC, CK are equal to the squares of FB, BK, of which the square of FC is equal to the square of FB, and therefore the remaining square of CK is equal to the remaining square of BK, and the straight line CK to the straight line BK: And because FB is equal to FC, and FK common to the triangles BFK, CFK, the two BF, FK are equal to the two CF, FK, each to each, and the base BK is equal to the base CK—therefore the angle BFK is equal to the angle CFK, and the angle BKF to the angle CKF, and therefore the angle BFC is double of the angle CFK, and the angle BKC of the angle CKF: For the like reason, the angle CFD is double of the angle CFL, and the angle CLD of the angle CLF: But, because the circumference BC is equal to the circumference CD, the angle BFC is equal to the angle CFD; and BFC is double of CFK, and CFD of CFL; therefore also the angle CFK is equal to the angle CFL: And the right angle FCK is equal to the right angle FCL: Therefore, in the two triangles FCK, FCL, there are two angles of one equal to two angles of the



other, each to each; and the side FC, which is adjacent to the equal angles in each, is common to both—therefore the other sides are equal (I. 26), each to each, and the third angle of the one to the third angle of the other; that is, CK is equal to CL, and the angle FKC to the angle FLC: And because CK is equal to CL, KL is double of CK: And in like manner it may be shewn that HK is double of BK: And because BK is equal to CK, as was demonstrated, and that HK is double of BK, and KL of CK, therefore also HK is equal to KL: And, in like manner, it may be shewn that GH, GM, ML are each of them equal to HK, or KL; therefore the pentagon GHKLM is equilateral: It is also equiangular: For, because, as has been shewn, the angle FKC is equal to the angle FLC, and the angle HKL is double of the angle FKC, and the angle KLM of the angle FLC, therefore the angle HKL is equal to the angle KLM: And, in like manner, it may be shewn that each of the angles KHG, HGM, GML is equal to the angle HKL or KLM; therefore the pentagon GHKLM is equiangular: and it has been shewn to be equilateral: Wherefore about the given circle has been described an equilateral and equiangular pentagon. Q.E.F.

PROP. XIII. PROB.

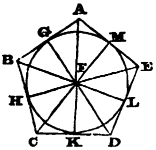
To inscribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon: it is required to inscribe a circle in the pentagon ABCDE.

Bisect the angles BCD, CDE by the straight lines

CF, DF, and from the point F, in which they meet, draw the straight lines FB, FA, FE.

Then, because BC is equal to CD, and CF common to the triangles BCF, DCF, the two sides BC, CF are equal to the two DC, CF, each to each, and the angle BCF is equal to the angle DCF—therefore the base BF is equal to the base FD, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle CBF is equal to the angle CDF: And because the angle CDE is double of the angle CDF, and that the angle CDE is equal to CBA, and the angle CDF to CBF, therefore the angle CBA is double of the angle CBF, that is, the angle ABC is bisected by the straight line BF: And in like manner it may be demonstrated, that the angles BAE, AED are bisected by the straight lines AF, EF.



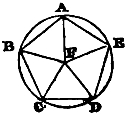
From the point F, draw FG, FH, FK, FL, FM perpendiculars on AB, BC, CD, DE, EA: Then, because the angle FCH is equal to the angle FCK, and the right angle FHC to the right angle FKC, therefore in the two triangles FHC, FKC there are two angles of one equal to two angles of the other, each to each, and the side FC, which is opposite to one of the equal angles in each, is common to both—therefore the other sides are equal, each to each, and therefore FH is equal to FK: And in like manner it may be demonstrated, that FL, FM, FG are each of them equal to FH, or FK; Therefore the five straight lines FG, FH, FK, FL, FM are equal to one another, and the circle described from the centre F, at the distance of any one of them, will pass through the extremities of the other four, and touch the straight lines AB, BC, CD, DE, EA, because the angles at the points G, H, K, L, M are right angles (3. 16): Wherefore a circle has been inscribed in the given equilateral and equiangular pentagon. Q. E. F.

PROP. XIV. PROB.

To describe a circle about a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon: it is required to describe a circle about it.

Bisect the angles BCD, CDE by the straight lines CF, DF, meeting in F, and join FB, FA, FE: Then it may be shewn, as in the preceding proposition, that the angles CBA, BAE, AED are bisected by the straight lines BF, AF, EF: And, because the angle BCD is equal to the angle CDE, and that FCD is the half of BCD, and CDF of CDE, therefore the angle FCD is equal to the angle FDC, and the side FC to the side FD: And in like manner it may be demonstrated that FB, FA, FE are each of them equal to FC or FD: Therefore the five straight lines FA, FB, FC, FD, FE are equal to one another, and the circle described from the centre F, at the distance of any one of them, will pass through the extremities of the other four, and be described about the equilateral and equiangular pentagon ABCDE. Q. E. F.

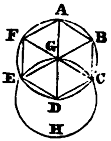


PROP. XV. PROB.

To inscribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given circle: it is required to inscribe an equilateral and equiangular hexagon in it.

Find the centre G of the circle ABCDEF, and draw the diameter AGD; and from the centre D, at the distance DG, describe the circle EGCH; join EG, CG, produce them to the points B, F, and join AB, BC, CD, DE, EF, FA: the hexagon ABCDEF is equilateral and equiangular.



For, because G is the centre of the circle ABCDEF, GE is equal to GD, and because D is the centre of the circle EGCH, DE is equal to DG; therefore GE is equal to DE, and the triangle EGD is equilateral, and therefore also equiangular (1. 5. *Cor.*): But the three angles of a triangle are together equal to two right angles; therefore the angle EGD is the third part of two right angles: And, in like manner, it may be shewn that the angle DGC is also the third part of two right angles: And because the straight line GC makes with EB the adjacent angles EGC, CGB together equal to two right angles, therefore the remaining angle CGB is also the third part of two right angles, and therefore the angles EGD, DGC, CGB are equal to one another: And to these are equal the vertical opposite angles BGA, AGF, FGE; therefore the six angles EGD, DGC, CGB, BGA, AGF, FGE are equal to one another: But equal angles stand upon equal circumferences; therefore the six circumferences AB, BC, CD, DE, EF, FA are equal to one another: And equal circumferences are subtended by equal straight lines; therefore the six straight lines are equal to one another, and the hexagon ABCDEF is equilateral: It is also equiangular: For, because the circumference AF is equal to the circumference

ED, to each of these add the circumference ABCD; therefore the whole circumference FABCD is equal to the whole circumference ABCDE: And the angle FED stands upon the circumference FABCD, and the angle AFE upon the circumference ABCDE; therefore the angle FED is equal to the angle AFE: And, in like manner, it may be demonstrated that the other angles of the hexagon ABCDEF are each of them equal to the angle AFE or FED; therefore the hexagon is equiangular; and it has been shewn to be equilateral: Wherefore, in the given circle, has been inscribed an equilateral and equiangular hexagon. Q. E. F.

Cor. From this it is manifest, that the side of the hexagon is equal to the straight line from the centre, that is, to the radius of the circle.

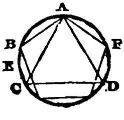
Also, if through the points A, B, C, D, E, F, there be drawn straight lines touching the circle, an equilateral and equiangular hexagon will be described about it, as may be demonstrated from what has been said of the pentagon: and likewise a circle may be inscribed in a given equilateral and equiangular hexagon, and circumscribed about it, by a method like to that used for the pentagon.

PROP. XVI. Prob.

To inscribe an equilateral and equiangular quindecagon in a given circle.

Let ABCD be the given circle: it is required to inscribe an equilateral and equiangular quindecagon in the circle ABCD.

Let AC be the side of an equilateral triangle inscribed in the circle (4. 2), and AB the side of an equilateral and



equiangular pentagon inscribed in the same (**4**. 11): Then, of such equal parts as the whole circumference ABCDF contains fifteen, the circumference ABC, being the third part of the whole, contains five, and the circumference AB, being the fifth part of the whole, contains three; therefore the circumference BC contains two of the same parts: Bisect BC in E (**3**. 30); then BE, EC are each of them the fifteenth part of the whole circumference: Therefore, if the straight lines BE, EC be drawn, and straight lines equal to them be placed round in the whole circle (**4**. 1), an equilateral and equiangular quindecagon will be inscribed in it. Q.E.F.

And, in the same manner as was done in the pentagon, if through the points of division, made by inscribing the quindecagon, straight lines be drawn touching the circle, an equilateral and equiangular quindecagon will be described about it: and likewise, as in the pentagon. a circle may be inscribed in a given equilateral and equiangular quindecagon, and circumscribed about it.

PROBLEMS.

BOOK IV.

1. IN a given circle place a line of given length, which shall pass through a given point.

2. Draw that diameter of a circle which shall pass at a given distance from a given point.

3. The square of the side of an equilateral triangle inscribed in a circle is triple the square of the side of the regular hexagon inscribed in the same circle.

4. An equilateral triangle is inscribed in a circle, and through the angular points tangents are drawn: shew that they will form an equilateral triangle, whose area is four times the former.

5. Shew that a circle may be described through the centres of four circles, each of which touches one side and the two adjacent sides produced of any quadrilateral.

6. If two circles be described, one without, the other within, a right-angled triangle, the sum of their diameters is equal to the sum of the sides containing the right angle.

7. The perpendicular from the vertex on the base of an equilateral triangle is equal to the side of an equilateral triangle inscribed in a circle whose diameter is the base.

8. Inscribe a square and a circle in a given quadrant.

9. Shew how to find the centres of the escribed circles of a triangle; and prove that the line joining any two of these centres is perpendicular to the line joining the centre of the inscribed circle with the angular point between them.

10. The line bisecting any angle of a triangle inscribed in a circle cuts the circumference in a point, which is equidistant from the extremities of the side opposite to the bisected angle and from the centre of the inscribed circle.

11. ABCD is a rectangle: if in the triangle ABC a circle be inscribed, touching AB, BC in E, F, and EGH, FGK be drawn parallel to AD, CD, then the rectangle KH is equal to the gnomon AFH.

12. The locus of the centres of the circles inscribed in all right-angled triangles on the same hypothenuse is the quadrant described on the hypothenuse.

13. In any triangle ABC, if AD, bisecting the angle A, cut BC in D, and from O, the centre of the inscribed circle, OE be drawn perpendicular to BC, shew that the angles BOE, COD are equal.

14. Inscribe a square in a given right-angled isosceles triangle.

15. Through two given points describe a circle touching a given circle: and shew that, of all lines that can be drawn from the two points to meet the convex circumference, those drawn to the points of contact thus obtained will contain the greatest possible angle.

16. The area of an inscribed regular hexagon is $\frac{3}{4}$ that of the one circumscribed about the same circle.

17. Upon a given line as diagonal describe a rhombus, so that two of its angles shall be double of the other two. Hence shew how a right angle may be trisected.

18. Given the vertical angle, the line drawn to the base bisecting that angle, and the difference between the base and the sum of the sides: construct the triangle.

19. Draw from the obtuse angle A of a given triangle to the base a line whose square shall be equal to the rectangle of the segments of the base.

20. The centres of the inscribed and circumscribed circles of an equilateral triangle coincide, and the diameter of one is double that of the other.

21. The line joining the centres of the inscribed and circumscribed circles of a triangle subtends at any one of the angular points an angle equal to the semidifference of the other two angles.

22. Given the angles of a triangle and the radius of the inscribed circle: construct the triangle.

23. Given the vertical angle of a triangle, and the radii of the inscribed and circumscribed circles: construct the triangle.

24. Within a given circle place eight circles, touching each other and the given circle.

25. The lines joining the alternate angles, or the intersections of the alternate sides of a regular pentagon, will form another regular pentagon.

26. Upon a given straight line describe a regular octagon.

27. Inscribe in a given circle a rectangle equal to a given rectilineal figure.

28. The area of a regular octagon inscribed in a circle is equal to the area of the rectangle of the sides of the inscribed and circumscribed squares.

29. Inscribe a square in the space included between two equal circles which cut each other.

30. Inscribe a circle in a given rhombus.

31. ABCD is a quadrilateral inscribed in a circle: produce AD, BC to meet in E; from any point F in DE draw FH parallel to BE meeting CD in H, and join FB cutting the circle in G: then shew that GH will cut the circle again in a fixed point K.

32. Inscribe an equilateral triangle in a given square, when the vertex is (i) in the middle of a side, (ii) in one angle.

33. Inscribe the least possible square in a given square.

34. Describe a circle which shall pass through one angle and touch two sides of a given square.

35. In the figure (4. 10), shew that AC is the side of a regular decagon inscribed in the larger circle, and of a regular pentagon inscribed in the smaller.

36. In the figure (4. 10) produce DC to meet the circle in F: then the angle ABF shall be triple of the angle BFD.

37. If ABCDE be a regular pentagon, shew that the angles ABE, BCA, CDB, DEC, EAD, are together equal to two right angles.

38. Given a regular pentagon: describe a triangle of the same area and altitude.

39. If two diagonals of a regular pentagon be drawn cutting one another, the larger segments will be each equal to a side of the pentagon.

40. Divide a right angle into five equal parts.

41. The opposite sides of a regular hexagon are parallel: and if any two sides of an inscribed hexagon are parallel to two other sides, the remaining two will also be parallel.

42. Inscribe a regular hexagon in a given equilateral triangle, and compare its area with that of the triangle.

43. Inscribe a regular dodecagon in a given circle, and shew that its area equals that of a square on the side of an equilateral triangle inscribed in the same circle.

44. Let AB, CD, two alternate sides of a regular polygon, be produced to meet in E: shew that the figure AECO can be inscribed in a circle, O being the centre of the polygon.

45. Given a regular polygon inscribed in a circle: shew how to describe another, with double the number of sides, in or about the same circle.

46. If R, r be the radii of circles described about and in a regular polygon, and R', r' the corresponding radii for a regular polygon of same perimeter and twice the number of sides, shew that $r' = \frac{1}{2}(R + r)$, and $R'^2 = Rr'$.

47. If from any point within a regular polygon of n sides perpendiculars be dropped on the sides, their sum will be n times the radius of the inscribed circle. How may this statement be made to include the case when the point is *without* the polygon?

48. If from the angles of an equilateral triangle, parallel lines, or *ordinates*, be drawn to any given line without it, their sum, and also the sum of the *abscissæ*, or segments of the line cut off by them, measured from a fixed point in it without them all, will be respectively three times the ordinate and abscissa of the centre of the inscribed circle. Shew that this is true of any regular polygon; and explain how the statement may be made to include the case of the line cutting the figure, and of the fixed point being any whatever in it.

49. If from the angles of an equilateral triangle ordinates be drawn to the diameter of the circumscribing circle, the single ordinate which falls on one side of the diameter will equal the sum of the two which fall on the other side; and so also the single segment of the diameter on one side of the centre will equal the sum of the two upon the other side.

50. If from any point P lines be drawn to the angles of an equilateral triangle ABC, shew that $PA^2 + PB^2 + PC^2$ is constant: and hence if ABCP, A′B′C′P′ be concentric circles, and ABC, A′B′C′ equilateral triangles inscribed in them, shew that

$$AP'^2 + BP'^2 + CP'^2 = A'P^2 + B'P^2 + C'P^2.$$

BOOK V.

DEFINITIONS.

I. A LESS magnitude is said to be a *part* of a greater magnitude, when the less measures the greater, that is, when the less is contained a certain number of times exactly in the greater.

II. A greater magnitude is said to be a *multiple* of a less, when the greater is measured by the less, that is, when the greater contains the less a certain number of times exactly.

III. Ratio is a mutual relation of two magnitudes of the same kind to one another in respect of quantity.

IV. Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

V. The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when, any equimultiples whatever of the first and third being taken, and any whatever of the second and fourth, if the multiple of the first be equal that of the second, the multiple of the third is also equal to that of the fourth, or, if greater, greater, or if less, less.

VI. Magnitudes which have the same ratio to one another are called proportionals.

When four magnitudes are proportionals, it is usually expressed by saying, the first is to the second as the third is to the fourth.

VII. When of the equimultiples of four magnitudes (taken as in the fifth definition), the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third has to the fourth, and the third is said to have to the fourth a less ratio than the first has to the second.

VIII. Analogy, or proportion, is the similitude of ratios.

IX. Proportion consists in three terms at least.

X. When three magnitudes are proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second.

XI. When four magnitudes are continued proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on, quadruplicate, &c., increasing the denomination still by unity, in any number of proportionals.

A. When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D be four maghitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratios of A to B, and B to C, and C to D.

And if A has to B the same ratio which E has to F, and B to C the same ratio that G has to H, and C to D the same that K has to L, then, by this definition, A is said to have to D the ratio compounded of ratios which are the same with the ratios of E to F, G to H, and K to L: and the same thing is to be understood when it is more briefly expressed by saying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D; then, for shortness' sake, M is said to have to N the ratio compounded of the ratios of E to F, G to H, and K to L.

XII. In proportionals, the antecedent terms are said to be homologous to one another, and so also are the consequents.

Geometers make use of the following technical words, to signify certain ways of changing either the order or magnitude of proportionals, so that they continue still to be proportionals:

XIII. *Permutando*, or *alternando*, by permutation or alternately; when there are four proportionals, and it is inferred, that the first is to the third as the second to the fourth; as is shewn in (**5**. 16):

XIV. *Invertendo*, by inversion; when there are four proportionals, and it is inferred, that the second is to the first as the fourth to the third (**5**. B):

XV. *Componendo*, by composition; when there are four proportionals, and it is inferred that the first, together with the second, is to the second, as the third, together with the fourth, is to the fourth (**5**. 18):

XVI. *Dividendo*, by division; when there are four proportionals, and it is inferred, that the excess of the first above the second is to the second, as the excess of the third above the fourth is to the fourth (**5**. 17):

XVII. *Convertendo*, by conversion; when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third to its excess above the fourth (**5**. E):

XVIII. *Ex æquali* (sc. *distantia*), or *ex æquo*, from equality of distance; when there is any number of magnitudes more than two, and as many others, such that they are proportionals when taken two and two of each rank, and it is inferred, that the first is to the last of the first rank of magnitudes, as the first is to the last of the others.

Of this there are the two following kinds, which arise from the different order in which the magnitudes are taken, two and two:

XIX. *Ex æquali*, or *ex æquali in proportione ordinatâ*, from equality in ordinate proportion; when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and the second to the third of the first rank, as the second to the third of the other; and so on in order; and the inference is made as mentioned in the preceding definition (**5**. 22):

XX. *Ex æquali in proportione perturbatâ seu inordinatâ*, from equality in perturbate or disorderly proportion; when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank; and the second is to the third of the first rank, as the last but two to the last but one of the second rank; and the third to the fourth of the first rank, as the last but three to the last but two of the second rank; and so on in a cross order; and the inference is made as before (**5**. 23).

AXIOMS.

I. Equimultiples of the same, or of equal magnitudes, are equal to one another.

II. Those magnitudes, of which the same or equal magnitudes are equimultiples, are equal to one another.

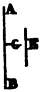
III. A multiple of a greater magnitude is greater than the same multiple of a less.

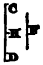
IV. That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

PROP. I. THEOR.

If any number of magnitudes be equimultiples of as many, each of each, whatever multiple any one of them is of its part, the same multiple shall all the first magnitudes be of all the other.

Let any number of magnitudes AB, CD be equimultiples of as many others E, F, each of each: whatever multiple AB is of E the same multiple shall AB and CD together be of E and F together.





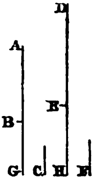
For, because AB is the same multiple of E that CD is of F, there are as many magnitudes in AB equal to E as there are in CD equal to F: Divide AB into magnitudes AG, GB, each equal to E, and CD into CH, HD, each equal to F: Then the number of the magnitudes AG, GB, will be the same as the number of the magnitudes CH, HD: And because AG is equal to E, and CH to F, therefore AG and CH together are equal to E and F together; and because GB is equal to E, and HD to F, therefore GB and HD together are equal to E and F together; and so on if there be any more magnitudes: Therefore, as many magnitudes as there are in AB equal to E, so many are there in AB and CD together, equal to E and F together, that is, whatever multiple AB is of E, the same multiple are AB and CD together of E and F together.

Wherefore, *If any number of magnitudes &c.* Q.E.D.

PROP. II. THEOR.

If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth, then shall the first together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth.

Let AB the first be the same multiple of C the second, that DE the third is of F the fourth, and let BG the fifth the same multiple of C the second, that EH the sixth is of F the fourth: then shall AG, the first together with the fifth, be the same multiple of C the second, that DH, the third together with the sixth, is of F the fourth.



For because AB is the same multiple of C that DE is of F, there are as many magnitudes in AB equal to C, as there are in DE equal to F; and, in like manner, there are as many magnitudes in BG equal to C, as there are in EH equal to F: Therefore, there are as many magnitudes in the whole AG equal to C, as there are in the whole DH equal to F, that is, AG, the first and fifth together, is the same multiple of C, the second, that DH, the third and sixth together, is of F, the fourth.

Wherefore, *If the first be the same multiple &c.* Q.E.D.

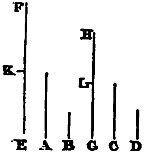
COR. In like manner it is plain, that, if *any* number of magnitudes be multiples of another, C, and as many magnitudes be the same multiples of another, F, each of each, then the whole of the first magnitudes is the same multiple of C, that the whole of the last magnitudes is of F.

PROP. III. THEOR.

If the first be the same multiple of the second which the third is of the fourth, and if of the first and third there be taken equimultiples, these shall be equimultiples, the one of the second, and the other of the fourth.

Let A the first be the same multiple of B the second, that C the third is of D the fourth, and of A and C let the equimultiples EF and GH be taken: then EF shall be the same multiple of B that GH is of D.

For because EF is the same multiple of A that GH is of C, there are as many magnitudes in EF equal to A, as there are in GH equal to C: Divide EF into magnitudes EK, KF, each equal to A, and GH into GL, LH, each equal to C: Then the number of the magnitudes EK, KF, will be the same as the number of the magnitudes GL, LH: And because A is the same multiple of B that C is of D, and that EK is equal to A, and GL to C, therefore EK is the same multiple of B that GL is of D: For the like reason, KF is the same multiple of B that LH is of D: Therefore because EK, the first, is the same multiple of B, the second, which GL, the third, is of D, the fourth, and FK, the fifth, is the same multiple of B, the second, which LH, the sixth, is of D, the fourth, therefore EF, the first together with the fifth, is the same multiple of B, the second, which GH, the third together with the sixth, is of D, the fourth (5. 2): And, in like manner, if there be more parts in EF and GH equal to A and C, it may be shewn by the help of (5. 2. *Cor.*) that EF is the same multiple of B which GH is of D.

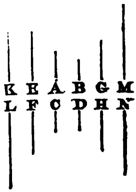


Wherefore, *If the first &c.* Q.E.D.

PROP. IV. Theor.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth, and if of the first and third there be taken any equimultiples whatever, and also any whatever of the second and fourth, then the multiple of the first shall have the same ratio to that of the second which the multiple of the third has to that of the fourth.

Let A, the first, have to B, the second, the same ratio which C, the third, has to D, the fourth, and of A and C let there be taken any equimultiples whatever E and F, and of B and D, any equimultiples whatever G and H: then E shall have the same ratio to G which F has to H.



Take of E and F any equimultiples whatever K and L, and of G and H, any equimultiples whatever M and N: Then, because E is the same multiple of A that F is of C, and of E and F have been taken equimultiples K and L, therefore K is the same multiple of A that L is of C (5. 3): For the like reason, M is the same multiple of B that N is of D: And because that A is to B as C is to D, and of A and C have been taken certain equimultiples K and L, and of B and D certain equimultiples M and N, therefore (5. *Def.* 5) if K be greater than M, L is greater than N, and if equal, equal, and if less, less: But K and L are any equimultiples whatever of E and F, and M and N are any equimultiples whatever of G and H; therefore (5. *Def.* 5) E is to G as F to H.

Wherefore, *If the first &c.* Q.E.D.

Cor. So also, if the first has the same ratio to the second which the third has to the fourth, then any equimultiples whatever of the first and third shall have the same ratio to the second and fourth, and, in like manner, the first and the third shall have the same ratio to any equimultiples whatever of the second and fourth.

Let A the first have to B the second, the same ratio which C, the third, has to D, the fourth, and of A and C, let there be taken any equimultiples whatever, E and F: then E shall be to B as F is to D.

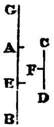
Take of E, F, any equimultiples whatever K, L, and of B, D, any equimultiples whatever G, H: Then it may be shewn, as before, that K is the same multiple of A that L is of C: And because A is to B as C is to D, and of A and C have been taken certain equimultiples K and L, and of B and D, certain equimultiples G and H, therefore, if K be greater than G, L is greater than H, and if equal, equal, and if less, less: But K and L are any equimultiples whatever of E and F, and G and H are any equimultiples whatever of B and D; therefore E is to B as F is to D. And in the same way the other case may be demonstrated.

PROP. V. Theor.

If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other, the remainder shall be the same multiple of the remainder that the whole is of the whole.

Let AB be the same multiple of CD, that AE, taken from the first, is of CF, taken from the other: the remainder EB shall be the same multiple of the remainder FD that the whole AB is of the whole CD.

Take AG the same multiple of FD that AE is of CF; therefore AE is the same multiple of CF that EG is of CD (5. 1): But AE is the same multiple of CF that AB is of CD (*Hyp.*); therefore EG is the same multiple of CD that AB is of CD, and therefore EG is equal to AB (5. *Ax.* 1): From each of these take AE, which is common to both; then the remainder AG is equal to the remainder EB: Therefore, because AE is the same multiple of CF that AG is of FD, and that AG is equal to EB, therefore AE is the same multiple of CF that EB is of FD: But AE is the same multiple of CF that AB is of CD; therefore EB is the same multiple of FD that AB is of CD.



Wherefore, *If one magnitude &c.* Q. E. D.

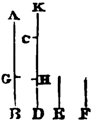
PROP. VI. THEOR.

If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two, the remainders shall be either equal to these others, or equimultiples of them.

Let the two magnitudes, AB, CD, be equimultiples of the two, E, F, and let AG, CH taken from the first two be equimultiples of the same E, F; the remainders GB, HD shall be either equal to E, F, or equimultiples of them.

First, let GB be equal to E; HD shall be equal to F.

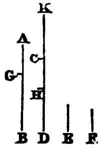
Make CK equal to F: Then, because AG is the same multiple of E that CH is of F, and that GB is equal to E and CK to F, therefore AB is the same multiple of E that KH is of F: But AB is the same multiple of E that CD is of F (*Hyp.*); therefore KH is the same multiple of F that CD is of



F, and therefore KH is equal to CD (5. *Ax.* 1): Take away the common magnitude CH from both; then the remainder KC is equal to the remainder HD: But KC is equal to F; therefore HD is equal to F.

Next, let GB be a multiple of E: HD shall be the same multiple of F.

Make CK the same multiple of F that GB is of E: Then, because AG is the same multiple of E that CH is of F, and GB the same multiple of E that CK is of F, therefore AB is the same multiple of E that KH is of F (5. 2): But AB is the same multiple of E that CD is of F; therefore KH is the same multiple of F that CD is of F, and therefore KH is equal to CD: Take away the common magnitude CH from both; therefore the remainder CK is equal to the remainder HD: And because CK is the same multiple of F that GB is of E, and that CK is equal to HD, therefore HD is the same multiple of F that GB is of E.



Wherefore, *If two magnitudes &c.* Q.E.D.

PROP. A. THEOR.

If the first of four magnitudes have to the second the same ratio which the third has to the fourth, then, if the first be greater than the second, the third is also greater than the fourth, and if equal, equal, and if less, less.

Take any equimultiples of each of them, as the doubles of each: Then (5. *Def.* 5), if the double of the first be greater than the double of the second, the double of the third is greater than the double of the fourth: But if the first be greater than the second, the double of the first is greater than the double of the second; therefore also the double of the third is greater than the double of the fourth, and therefore the third is greater than the fourth: And, in like manner, if the first be equal to the second, or less than it, the third may be proved to be equal to the fourth, or less than it.

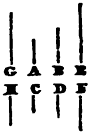
Wherefore, *If the first &c.* Q. E. D.

PROP. B. Theor.

If four magnitudes are proportionals, they are proportionals also when taken inversely.

If A be to B as C is to D, then also, inversely, B shall be to A as D to C.

Take of B, D, any equimultiples whatever, E, F, and of A, C, any equimultiples whatever G, H: Then, first, if E be greater than G, G is less than E: And because A is to B as C is to D, and of A and C, the first and third, G and H are equimultiples, and of B and D, the second and fourth, E and F are equimultiples, and that G is less than E, therefore also H is less than F, that is, F is greater than H: If therefore E be greater than G, F is greater than H; and, in like manner, if E be equal to G, F may be shewn to be equal to H, and if less, less: But E and F are any equimultiples whatever of B and D, and G and H any whatever of A and C; therefore, B is to A as D to C.



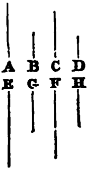
Wherefore, *If four magnitudes &c.* Q. E. D.

PROP. C. Theor.

If the first be the same multiple of the second, or the same part of it, that the third is of the fourth, the first is to the second as the third is to the fourth.

First, let A be the same multiple of B that C is of D: A shall be to B as C is to D.

Take of A, C, any equimultiples whatever E, F, and of B, D, any equimultiples whatever G, H: Then, because A is the same multiple of B that C is of D, and that E is the same multiple of A that F is of C, therefore E is the same multiple of B that F is of D (5. 3), that is, E and F are equimultiples of B and D: But G and H are equimultiples of B and D; therefore, if E be a greater multiple of B than G is of B, F is a greater multiple of D than H is of D; that is, if E be greater than G, F is greater than H: And, in like manner, if E be equal to G, F may be shewn to be equal to H, and if less, less: But E, F are any equimultiples whatever of A, C, and G, H are any equimultiples whatever of B, D; therefore A is to B as C is to D.



Next, let A be the same part of B that C is of D: A shall be to B, as C is to D.

For B is the same multiple of A that D is of C: Therefore, by the preceding case, B is to A as D is to C; and, inversely (5. B), A is to B as C is to D.

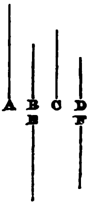
Wherefore, *If the first be the same multiple &c.* Q.E.D.

PROP. D. THEOR.

If the first be to the second as the third to the fourth, and if the first be a multiple, or a part of the second, the third shall be the same multiple, or the same part of the fourth.

Let A be to B as C is to D, and first, let A be a multiple of B: C shall be the same multiple of D.

Take E equal to A, and, whatever multiple A or E is of B, make F the same multiple of D: Then, because A is to B as C is to D, and of B and D have been taken equimultiples, E and F, therefore, A is to E as C to F (5. *Cor.* 4): But A is equal to E; therefore C is equal to F (5. A): And F is the same multiple of D that A is of B; therefore also C is the same multiple of D that A is of B.



Next, let A be a part of B: C shall be the the same part of D.

For, because A is to B as C is to D, therefore, inversely (5. B), B is to A as D to C: But A is a part of B; therefore B is a multiple of A, and, therefore, by the preceding case, D is the same multiple of C—that is, C is the same part of D that A is of B.

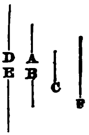
Wherefore, *If the first &c.* Q.E.D.

PROP. VII. THEOR.

Equal magnitudes have the same ratio to the same magnitudes: and the same has the same ratio to equal magnitudes.

Let A and B be equal magnitudes, and C any other: A and B shall have each the same ratio to C, and C shall have the same ratio to each of A and B.

Take of A and B any equimultiples whatever D and E, and of C any multiple whatever F: Then, because D is the same multiple of A that E is of B, and that A is equal to B, therefore D is equal to E: Hence, if D be greater than F, E will be greater than F, and if equal, equal, and if less, less: But D and E are any equimultiples whatever of A and B, and F is any multiple of C; therefore, A is to C as B is to C.



So also C shall have the same ratio to A that it has to B.

For the same construction being made, it may be shewn, as before, that D is equal to E: therefore, if F be greater than D, F is also greater than E, and if equal, equal, and if less, less: But F is any multiple whatever of C, and D and E are any equimultiples whatever of A and B; therefore, C is to A as C is to B.

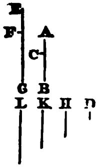
Wherefore, *Equal magnitudes &c.* Q.E.D.

PROP. VIII. THEOR.

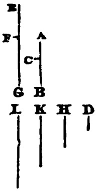
Of unequal magnitudes the greater has a greater ratio to the same than the less has: and the same magnitude has a greater ratio to the less than it has to the greater.

Let AB, BC be unequal magnitudes, of which AB is the greater, and let D be any magnitude whatever: AB shall have to D a greater ratio than BC has to D, and D shall have to BC a greater ratio than it has to AB.

If the magnitude which is not the greater of the two AC, CB, be not less than D, take EF, FG, the doubles of AC, CB, (fig. 1): But if that which is not the greater of the two AC, CB, be less than D (fig. 2), it can be multiplied, so as to become greater than D: Let it be so multiplied, and let the other be multiplied as often; and let EF, FG be the equimultiples thus taken of AC, CB, so that EF, FG, will be each of them greater than D: And in each case, take H the double of D, K its triple, and so on, till we come to that multiple of D which first becomes greater than FG: Let L be that multiple, and K the multiple of D which is next less than L.



Then, because L is the multiple of D, which first becomes greater than FG, the next preceding multiple K is not greater than FG, that is, FG is not less than K: And because EF is the same multiple of AC that FG is of CB, therefore EG is the same multiple of AB that FG is of CB (**5.** 1), that is, EG, FG, are equimultiples of AB, CB: And it was shewn that FG is not less than K, and, by construction, EF is greater than D; therefore the whole EG is greater than K and D together: But K and D together are equal to L; therefore EG is greater than L: But FG is not greater than L; and EG, FG are equimultiples of AB, BC, and L is a multiple of D; therefore (**5.** *Def.* 7) AB has to D a greater ratio than BC has to D.



Also, D shall have to BC a greater ratio than it has to AB.

For, the same construction being made, it may be shewn, in like manner, that L is greater than FG, but not greater than EG: And L is a multiple of D; and EG, FG are equimultiples of AB, BC; therefore D has to BC a greater ratio than it has to AB.

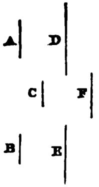
Wherefore, *Of unequal magnitudes &c.* Q.E.D.

PROP. IX. THEOR.

Magnitudes, which have the same ratio to the same magnitude, are equal to one another: and those, to which the same magnitude has the same ratio, are equal to one another.

First, let A, B have each the same ratio to C: A shall be equal to B.

For, if A be not equal to B, one of them is greater than the other; let A be the greater: Then, as in (**5.** 8), there are some equimultiples of A and B, and some multiple of C, such that the multiple of A is greater than the multiple of C, but the multiple of B not greater than that of C: Let such multiples be taken; and let D, E be the equimultiples thus taken of A, B, and F the multiple of C, so that D may be greater than F, but E not greater than F: Then, because A is to C as B is to C, and of A, B, are taken equimultiples D, E, and of C a multiple F, and that D is greater than F, therefore also E is greater than F (5. *Def.* 5): But E is not greater than F—which is impossible: Therefore A is not unequal to B, that is, they are equal.



Next, let C have the same ratio to each of A and B: A shall be equal to B.

For, if A be not equal to B, one of them is greater than the other; let A be the greater: Then, as in (5. 8), there is some multiple F of C, and there are some equimultiples D and E of A and B, such that F is greater than E, but not greater than D: Now, because C is to A as C is to B, and that F is a multiple of C, and D, E, equimultiples of A, B, and that F is greater than E, therefore also F is greater than D: But F is not greater than D—which is impossible: Therefore A is not unequal to B; that is, they are equal.

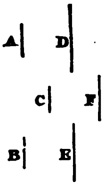
Wherefore, *Magnitudes which &c.* Q.E.D.

PROP. X. THEOR.

That magnitude, which has a greater ratio than another has to the same magnitude, is the greater of the two: and that magnitude, to which the same has a greater ratio than it has to another magnitude, is the lesser of the two.

First, let A have to C a greater ratio than B has to C: A shall be greater than B.

For, because A has a greater ratio to C than B has to C, there are (5. *Def.* 7) some equimultiples of A and B and some multiple of C, such that the multiple of A is greater than the multiple of C, but the multiple of B not greater than the multiple of C: Let such multiples be taken; and let D, E be the equimultiples of A, B, and F the multiple of C, such that D is greater than F, but E not greater than F, and therefore D is greater than E: But D and E are equimultiples of A and B; therefore (5. *Ax.* 4) A is greater than B.



Next, let C have to B a greater ratio than it has to A: B shall be less than A.

For there is some multiple F of C, and there are some equimultiples D and E of A and B, such that F is greater than E, but not greater than D, and therefore E is less than D: But D and E are equimultiples of A and B; therefore B is less than A.

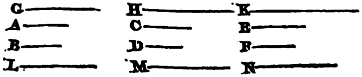
Wherefore, *That magnitude &c.* Q.E.D.

PROP. XI. THEOR.

Ratios that are the same to the same ratio are the same to one another.

Let A be to B as C is to D, and let C be to D as E is to F: then A shall be to B as E is to F.

Take of A, C, E, any equimultiples whatever G, H, K,



and of B, D, F, any equimultiples whatever L, M, N: Then, because A is to B as C is to D, and that G, H are equimultiples of A, C, and L, M, of B, D, therefore if G be greater than L, H is greater than M, and if equal, equal, and if less, less: Again, because C is to D as E is to F, and H, K are equimultiples of C, E, and M, N, of D, F, therefore if H be greater than M, K is greater than N, and if equal, equal, and if less, less: But it has been shewn that, if G be greater than L, H is greater than M, and if equal, equal, and if less, less; therefore if G be greater than L, K is greater than N, and if equal, equal, and if less, less: And G, K are any equimultiples whatever of A, E, and L, N any whatever of B, F; therefore, A is to B as E is to F.

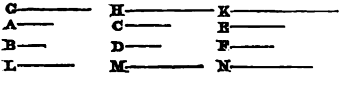
Wherefore, *Ratios that &c.* Q.E.D.

PROP. XII. THEOR.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

Let any number of magnitudes A, B, C, D, E, F, be proportionals, that is, as A is to B, so let C be to D, and E to F: then, as A is to B, so shall A, C, E together be to B, D, F together.

Take of A, C, E any equimultiples whatever G, H, K,



and of B, D, F any equimultiples whatever L, M, N: Then, because A is to B as C is to D and as E to F, and that G, H, K are equimultiples of A, C, E, and L, M, N equimultiples of B, D, F, therefore, if G be greater than

L, H will be greater than M and K greater than N, and if equal, equal, and if less, less: Therefore, if G be greater than L, G, H, K together will be greater than L, M, N together, and if equal, equal, and if less, less: But G, and G, H, K together, are any equimultiples of A, and A, C, E together, for (5. 1) if there be any number of magnitudes equimultiples of as many, each of each, whatever multiple one of them is of its part, the same multiple will the whole be of the whole; and for the same reason L, and L, M, N are any equimultiples of B, and B, D, F: Therefore as A is to B so are A, C, E together to B, D, F together.

Wherefore, *If any number &c.* Q.E.D.

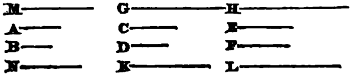
PROP. XIII. THEOR.

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth, the first shall also have to the second a greater ratio than the fifth has to the sixth.

Let A the first have the same ratio to B the second which C the third has to D the fourth, but let C the third have to D the fourth a greater ratio than E the fifth has to F the sixth: then also the first A shall have to the second B a greater ratio than the fifth E to the sixth F.

For, because C has a greater ratio to D than E to F, there are some equimultiples of C and E, and some of D and F, such that the multiple of C is greater than the multiple of D, but the multiple of E not greater than the multiple of F (5. *Def.* 7): Let such multiples be taken, and let G, H, be equimultiples of C, E, and K, L, equimultiples of D, F, such that G may be greater than

K, but H not greater than L: Also whatever multiple G is of C, take M the same multiple of A; and whatever multiple K is of D, take N the same multiple of B: Then, because A is to B as C to D, and M and G are equimultiples of A and C, and N and K of B and D, therefore if M be greater than N, G is greater than K, and if equal, equal, and if less, less: But G is greater than K; therefore M is greater than N: And H is not greater than L; and M, H are equimultiples of A, E, and N, L equimultiples of B, F; therefore (5. *Def.* 7) A has a greater ratio to B than E has to F.



Wherefore, *If the first &c.* Q.E.D.

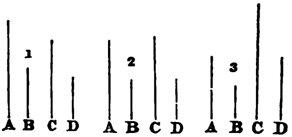
COR. Also, if the first have a greater ratio to the second than the third has to the fourth, but the third the same ratio to the fourth which the fifth has to the sixth, it may be shewn, in like manner, that the first has a greater ratio to the second than the fifth has to the sixth.

PROP. XIV. THEOR.

If the first has the same ratio to the second which the third has to the fourth, then, if the first be greater than the third, the second shall be greater than the fourth, and if equal, equal, and if less, less.

Let the first A have the same ratio to the second B which the third C has to the fourth D: then, if A be greater than C, B shall be greater than D, if equal, equal, and if less, less.

First, let A be greater than C: Then, since A is greater than C, and B any other magnitude, therefore A has to B a greater ratio than C to B (5. 8): But, as A is to B so is C to D; therefore also C has to D a greater ratio than C has to B (5. 13): But, of two magnitudes, that to which the same has the greater ratio is the lesser (5. 10); therefore D is less than B, that is, B is greater than D.



Next, let A be equal to C: Then, A is to B as C, that is, A, is to D; therefore (5. 9) B is equal to D.

Lastly, let A be less than C: Then C is greater than A; and because C is to D as A is to B, and that C is greater than A, therefore, by the first case, D is greater than B, that is, B is less than D.

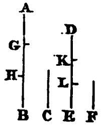
Wherefore, *If the first &c.* Q. E. D.

PROP. XV. THEOR.

Magnitudes have the same ratio to one another which their equimultiples have.

Let AB and DE be equimultiples of C and F: C shall be to F as AB to DE.

For, because AB is the same multiple of C that DE is of F, there are as many magnitudes in AB equal to C, as there are in DE equal to F: Divide AB into magnitudes, each equal to C, viz. AG, GH, HB, and DE into magnitudes, each equal to F, viz. DK, KL, LE: Then the number of the magnitudes AG, GH, HB, will be equal to the number of the magnitudes, DK, KL, LE: And because AG,



GH, HB are all equal, and that DK, KL, LE are also equal to one another, therefore AG is to DK as GH to KL, and as HB to LE: But as one of the antecedents is to its consequent, so are all the antecedents together to all the consequents together (5. 12); therefore, as AG is to DK so is AB to DE: But AG is equal to C, and DK to F; therefore, as C is to F so is AB to DE.

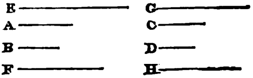
Wherefore, *Magnitudes &c.* Q. E. D.

PROP. XVI. THEOR.

If four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.

Let the four magnitudes A, B, C, D be proportionals, that is, let A be to B as C to D: they shall also be proportionals when taken alternately, that is, A shall be to C as B to D.

Take of A, B, any equimultiples whatever E, F, and of C, D, any equimultiples whatever G, H: Then, because E is the same multiple of A that F is of B, and that magnitudes have the same ratio to one another which their equimultiples have (5. 15) therefore A is to B as E is to F: But A is to B as C is to D; therefore C is to D as E is to F (5. 11): Again, because G, H are equimultiples of C, D, therefore C is to D as G is to H: But C is to D as E is to F; therefore E is to F as G to H: But when four magnitudes are proportionals, if the first be greater than the third, the second shall be greater than the fourth, and if equal, equal, and if less, less (5. 14); therefore, if E be greater than G, F is greater than H, and if equal, equal, and if less, less: But E, F, are any equimultiples whatever



of A, B, and G, H any whatever of C, D; therefore A is to C as B is to D.

Wherefore, *If four magnitudes &c.* Q.E.D.

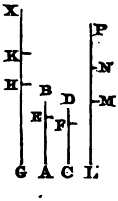
COR. Hence, by the help of (5. 14), it follows that, if four magnitudes be proportionals, then if the first be greater than the second, the third will be greater than the fourth, and if equal, equal, and if less, less.

PROP. XVII. THEOR.

If four magnitudes, taken jointly, be proportionals, they shall also be proportionals when taken separately; that is, *if two magnitudes together have to one of them the same ratio which two others have to one of these, the remaining one of the first two shall have to the other the same ratio which the remaining one of the last two has to the other of these.*

Let AB, BE, CD, DF be four magnitudes which, taken jointly, are proportionals, that is, let AB be to BE as CD to DF: they shall also be proportionals when taken separately, that is, AE shall be to EB as CF to FD.

Take of AE, EB, CF, FD, any equimultiples whatever GH, HK, LM, MN, and, again, of EB, FD, take any equimultiples whatever KX, NP: Then, because GH is the same multiple of AE that HK is of EB, therefore GH is the same multiple of AE that GK is of AB (5.1): But GH is the same multiple of AE that LM is of CF; therefore GK is the same multiple of AB that LM is of CF: Again, because LM is the same multiple of CF that MN is of FD, therefore LM is the same multiple of CF that LN is of CD: But LM was shewn to be the same multiple of CF that GK is of AB; therefore GK is the same multiple of AB that LN is of CD, that



is, GK, LN, are equimultiples of AB, CD: Again, because HK is the same multiple of EB that MN is of FD, and that KX is also the same multiple of EB that NP is of FD, therefore HX is the same multiple of EB that MP is of FD (5.2), that is, HX, MP, are equimultiples of EB, FD: And because AB is to BE as CD is to DF, and that GK, LN, are equimultiples of AB, CD, and HX, MP, are equimultiples of EB, FD, therefore, if GK be greater than HX, LN is greater than MP, and if equal, equal, and if less, less: But if GH be greater than KX, then, by adding HK to both, GK is greater than HX; therefore also LN is greater than MP, and, by taking away MN from both, LM is greater than NP; that is, if GH be greater than KX, LM is greater than NP: And, in like manner, it may be shewn that, if GH be equal to KX, LM is equal to NP, and if less, less: But GH, LM, are any equimultiples whatever of AE, CF, and KX, NP are any whatever of EB, FD; therefore AE is to EB as CF to FD.

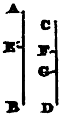
Wherefore, *If four magnitudes &c.* Q.E.D.

PROP. XVIII. THEOR.

If four magnitudes, taken separately, be proportionals, they shall also be proportionals when taken jointly: that is, *if the first be to the second, as the third to the fourth, the first and second together shall be to the second, as the third and fourth together to the fourth.*

Let the four magnitudes AE, EB, CF, FD be proportionals, that is, let AE be to EB as CF to FD: they shall also be proportionals, when taken jointly, that is, AB shall be to BE as CD to DF.

For, if not, let AB be to BE as CD to DG: Then (5. 17) since these four magnitudes are proportionals, they are also proportionals, when taken separately, that is, AE is to EB as CG to GD: But (*Hyp.*) AE is to EB as CF to FD; therefore CF is to FD as CG to GD: Therefore (5. 14) if CF be greater than CG, FD will be greater than GD, or if less, less: But, since CF and FD together make up the same whole as CG and GD, therefore, if CF be greater than CG, FD will be less than GD, or if less, greater—which is absurd: therefore AB is to BE as CD to DF.



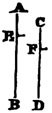
Wherefore, *If four magnitudes &c.* Q.E.D.

PROP. XIX. Theor.

If a whole magnitude be to a whole as a magnitude taken from the first is to a magnitude taken from the other, then the remainder shall be to the remainder as the whole to the whole.

Let the whole AB, be to the whole CD, as AE, a magnitude taken from AB, is to CF, a magnitude taken from CD: then the remainder EB shall be to the remainder FD as the whole AB to the whole CD.

For, because AB is to CD as AE to CF, therefore, alternately (5. 16), AB is to AE as CD to CF: But, if magnitudes taken jointly be proportionals, they are also proportionals when taken separately (5. 17); therefore, EB is to AE as FD to CF, and, alternately, EB is to FD as AE to CF, that is, as AB to CD (*Hyp.*).



Wherefore, *If the whole &c.* Q.E.D.

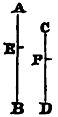
Cor. If the whole be to the whole as a magnitude taken from the first is to a magnitude taken from the other, then also the remainder shall be to the remainder as the magnitude taken from the first is to that taken from the other; that is, if AB be to CD as EB to FD, then AE shall be to CF as EB to FD.

PROP. E. THEOR.

If four magnitudes be proportionals, they shall also be proportionals by conversion; that is, *the first shall be to its excess above the second as the third to its excess above the fourth.*

Let AB be to BE as CD to DF: then BA shall be to AE as DC to CF.

For, because AB is to BE as CD to DF, therefore, by division (5. 17), AE is to EB as CF to FD, and, by inversion (5. B), BE is to EA as DF to FC: Therefore, by composition (5. 18), BA is to AE as DC is to CF.



Wherefore, *If four magnitudes &c.* Q.E.D.

PROP. XX. THEOR.

If there be three magnitudes, and other three, which, taken two and two, have the same ratio, then, if the first be greater than the third, the fourth shall be greater than the sixth, and if equal, equal, and if less, less.

Let A, B, C be three magnitudes, and D, E, F other three, which, taken two and two, have the same ratio, that is, let A be to B as D to E, and let B be to C as E to F: then, if A be greater than C, D shall be greater than F, and if equal, equal, and if less, less.

First, let A be greater than C; D shall be greater than F: For, because A is greater than C, and B is any other magnitude, therefore (5. 8) A has to B a greater ratio than C has to B: But A is to B as D is to E; therefore D has to E a greater ratio than C to B: And because B is to C as E to F, by inversion, C is to B as F is to E: And D was shewn to have to E a greater ratio than C to B; therefore D has to E a greater ratio than F to E, and therefore (5. 10) D is greater than F.

Next, let A be equal to C: D shall be equal to F: For, because A is equal to C, and B is any other magnitude, therefore A is to B as C is to B (5. 7): But A is to B as D to E, and C is to B as F to E; therefore D is to E as F to E, and therefore D is equal to F (5. 9).

Lastly, let A be less than C; D shall be less than F: For C is greater than A; and, as before, C is to B as F to E, and, in like manner, B is to A as E to D; therefore, by the first case, F is greater than D, that is, D is less than F.

Wherefore, *If there be three &c.* Q. E. D.

PROP. XXI. THEOR.

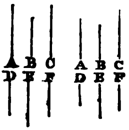
If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order, then if the first magnitude be greater than the third, the fourth shall be greater than the sixth, and if equal, equal, and if less, less.

Let A, B, C be three magnitudes, and D, E, F other three, which have the same ratio, taken two and two, but in a cross order, that is, let A be to B as E is to F, and let B be to C as D is to E: then if A be greater than C, D shall be greater than F, and if equal, equal, and if less, less.

First, let A be greater than C; D shall be greater than F: For, because A is greater than C, and B is any other magnitude, therefore A has to B a greater ratio than C has to B: But A is to B as E to F; therefore E has to F a greater ratio than C to B: And because B is to C as D to E, therefore, by inversion, C is to B as E to D: And E was shewn to have to F a greater ratio than C to B; therefore E has to F a greater ratio than E to D; and therefore F is less than D, that is, D is greater than F.

Next, let A be equal to C; D shall be equal to F: For, because A is equal to C, and B is any other magnitude, therefore A is to B as C is to B: But A is to B as E to F, and C is to B as E to D; therefore E is to F as E to D, and therefore D is equal to F.

Lastly, let A be less than C; D shall be less than F: For C is greater than A; and, as before, C is to B as E to D, and, in like manner, B is to A as F to E; therefore, by the first case, since C is greater than A, F is greater than D, that is, D is less than F.



Wherefore, *If there be three &c.* Q. E. D.

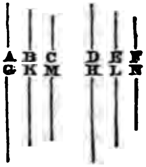
PROP. XXII. THEOR.

If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio, then the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.

N.B. This is usually cited by the words "ex æquali."

First, let there be three magnitudes, A, B, C, and other three, D, E, F, which, taken two and two, have the same ratio, that is, let A be to B as D to E, and B to C as E to F: then A shall be to C as D to F.

Take of A, D, any equimultiples whatever G, H, and of B, E, any whatever K, L, and of C, F any whatever M, N: Then, because A is to B as D to E, and that G, H are equimultiples of A, D, and K, L equimultiples of B, E, therefore G is to K as H to L (5. 4): And, for the like reason, K is to M as L to N: And because there are three magnitudes G, K, M, and other three H, L, N, which, taken two and two, have the same ratio, therefore (5. 20) if G be greater than M, H is greater than N, and if equal, equal, and if less, less: But G, H are any equimultiples whatever of A, D, and M, N are any equimultiples whatever of C, F; therefore A is to C as D to F.



Next, let there be four magnitudes, A, B, C, D, and other four E, F, G, H, which, taken two and two, have the same ratio, that is, let A be to B as E to F, B to C as F to G, and C to D as G to H: then A shall be to D as E to H.

A. B. C. D.
E. F. G. H.

For, because A, B, C are three magnitudes, and E, F, G other three, which, taken two and two, have the same ratio, therefore, by the first case, A is to C as E to G: But C is to D as G is to H; therefore also, by the first case, A is to D as E to H: And so we may proceed, whatever be the number of magnitudes.

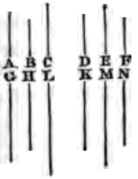
Wherefore, *If there be any number &c.* Q. E. D.

PROP. XXIII. THEOR.

If there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same *ratio, then the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.*

First, let there be three magnitudes A, B, C, and other three D, E, F, which, taken two and two in a cross order, have the same ratio, that is, let A be to B as E to F, and B to C as D to E: then A shall be to C as D to F.

Take of A, B, D, any equimultiples whatever G, H, K, and of C, E, F, any equimultiples whatever L, M, N: Then, because G, H are equimultiples of A, B, therefore (5. 15) A is to B as G to H: And for the like reason, E is to F as M to N: But A is to B as E to F; therefore G is to H as M to N: And because B is to C as D to E, and that H, K are equimultiples of B, D, and L, M equimultiples of C, E, therefore (5. 4) H is to L as K to M: And it has been shewn that G is to H as M to N: So then, because there are three magnitudes G, H, L, and other three K, M, N, which, taken two and two in a cross order, have the same ratio, therefore (5. 21) if G be greater than L, K is greater than N, and if equal, equal, and if less, less: But G, K are any equimultiples whatever of A, D, and L, N any whatever of C, F; therefore A is to C as D to F.



Next, let there be four magnitudes A, B, C, D, and other four E, F, G, H, which, taken two and two in a cross order, have the same ratio; that is, let A be to B as G to H, B to C as F to G, and C to D as E to F: then A shall be to D as E to H.

A. B. C. D.
E. F. G. H.

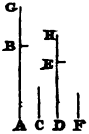
For, because A, B, C are three magnitudes, and F, G, H other three, which, taken two and two in a cross order, have the same ratio, therefore, by the first case, A is to C as F to H: But C is to D as E to F; therefore also, by the first case, A is to D as E to H: and so we may proceed, whatever be the number of magnitudes.

Wherefore, *If there be any number &c.* Q.E.D.

PROP. XXIV. Theor.

If the first has to the second the same ratio which the third has to the fourth, and the fifth to the second the same ratio which the sixth has to the fourth, then the first and fifth together shall have to the second the same ratio which the third and sixth together have to the fourth.

Let AB the first have to C the second the same ratio which DE the third has to F the fourth, and let BG the fifth have to C the second the same ratio which EH the sixth has to F the fourth: then AG, the first and fifth together, shall have the same ratio to C the second which DH, the third and sixth together, has to F the fourth.



For, because BG is to C as EH to F, therefore, by inversion, C is to BG as F to EH: And, because AB is to C as DE to F, and C is to BG as F to EH, therefore, ex æquali (5. 22), AB is to BG as DE to EH: And therefore jointly (5. 18) AG is to BG as DH to EH: But BG is to C as HE to F; therefore, ex æquali, AG is to C as DH to F.

Wherefore, *If the first &c.* Q.E.D.

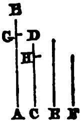
Cor. 1. On the same hypothesis, the excess of the first and fifth shall be to the second as the excess of the third and sixth to the fourth. The demonstration of this is the same with that of the proposition, if division be used instead of composition.

Cor. 2. The proposition holds true of two ranks of magnitudes, whatever be their number, of which each of the first rank has the same ratio to the second magnitude that the corresponding one of the second rank has to a fourth magnitude—as is manifest.

PROP. XXV. Theor.

If four magnitudes of the same kind be proportionals, the greatest and least of them together shall be greater than the other two together.

Let AB be to CD as E to F, and let AB be the greatest of them, and consequently F the least (5. A. & 14): then AB and F together shall be greater than CD and E together.



Take AG equal to E, and CH equal to F: Then, because AB is to CD as E to F, and that AG is equal to E, and CH to F, therefore AB is to CD as AG to CH, and therefore also (5. 19) the remainder BG is to the remainder DH as AB to CD: But AB is greater than CD; therefore BG is greater than DH (5. A.): And because AG is equal to E, and CH to F, therefore AG and F together are equal to CH and E together: And if to the unequal magnitudes BG, DH, of which BG is the greater, there be added equal magnitudes, viz. AG and F to BG, and CH and E to DH, then AB and F together are greater than CD and E.

Wherefore, *If four magnitudes &c.*

BOOK VI.

DEFINITIONS.

I. SIMILAR rectilineal figures are those which have their angles equal, each to each, and the sides about the equal angles proportionals.

II. Reciprocal figures, viz. triangles and parallelograms, are such as have their sides about two of their angles proportionals in such a manner, that a side of the first figure is to a side of the other, as the remaining side of this other is to the remaining side of the first.

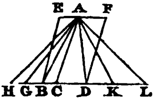
III. A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment as the greater segment is to the less.

IV. The altitude of any figure is the straight line drawn from its vertex perpendicular to the base.

PROP. I. THEOR.

Triangles and parallelograms of the same altitude are to one another as their bases.

Let the triangles ABC, ACD, and the parallelograms EC, CF, have the same altitude, viz. the perpendicular drawn from the vertex A to BD: then, as the base BC is to the base CD, so shall the triangle ABC be to the triangle ACD, and the parallelogram EC to the parallelogram CF.



Produce BD both ways, and take any number of straight lines BG, GH, each equal to BC, and, any number, DK, KL, each equal to CD; and join AG, AH, AK, AL.

Then, because CB, BG, GH are all equal, the triangles ABC, AGB, AHG, are all equal; and therefore, whatever multiple the base HC is of the base BC, the same is the triangle AHC of the triangle ABC: For the like reason, whatever multiple the base CL is of the base CD, the same is the triangle ACL of the triangle ACD: And, if the base HC be equal to the base CL, the triangle AHC is equal to the triangle ACL, and if greater, greater, and if less, less: Therefore, since there are four magnitudes, viz. the two bases BC, CD, and the two triangles ABC, ACD, and of the first and third, viz. the base BC and the triangle ABC, have been taken any equimultiples whatever, viz. the base HC and the triangle AHC, and of the second and fourth, viz. the base CD and triangle ACD, have been taken any equimultiples whatever, viz. the base CL and triangle ACL, and that it has been shewn that, if the base HC be greater than the base CL, the triangle AHC is greater than the triangle ACL, and if equal, equal, and if less, less—therefore (**5**. *Def.* 5), as the base BC is to the base CD, so is the triangle ABC to the triangle ACD.

And, because the parallelogram CE is double of the triangle ABC, and the parallelogram CF double of the triangle ACD, and that magnitudes have the same ratio which their equimultiples have (**5**. 15), therefore, the triangle ABC is to the triangle ACD as the parallelogram EC to the parallelogram CF: But it has been shewn that the triangle ABC is to the triangle ACD as the base BC is to the base CD; therefore (**5**. 11), as the base BC is to the base CD, so is the parallelogram EC to the parallelogram CF.

Wherefore, *Triangles &c.* Q. E. D.

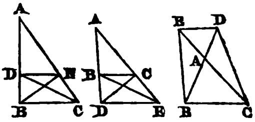
COR. Triangles and parallelograms of *equal* altitudes are to one another as their bases.

PROP. II. THEOR.

If a straight line be parallel to the base of a triangle, it shall cut the sides, or the sides produced, proportionally: and if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section shall be parallel to the base.

Let DE be parallel to BC, the base of the triangle ABC: BD shall be to DA as CE to EA.

Join BE, CD: Then the triangle BDE is equal to the triangle CDE, because they are on the same base DE, and between the same parallels DE, BC: And ADE is another triangle; therefore (**5**. 7) the triangle BDE is to the triangle ADE as the triangle CDE is to the triangle ADE: But the triangle BDE is to the triangle ADE as BD is to DA (**6**. 1), because they have the same altitude, viz. the perpendicular from E on AB; and so also the triangle CDE is to the triangle ADE as CE to EA: Therefore BD is to DA as CE to EA.



Next, let BD be to DA as CE to EA, and join DE: then DE shall be parallel to BC.

For, the same construction being made, because BD is to DA as CE to EA, and as BD to DA so is the triangle BDE to the triangle ADE, and as CE to EA so is the triangle CDE to the triangle ADE--therefore the triangle BDE is to the triangle ADE as the triangle CDE to the triangle ADE, and therefore (5. 9) the triangle BDE is equal to the triangle CDE: But they are on the same base DE; therefore DE is parallel to BC.

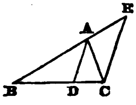
Wherefore, *If a straight line &c.* Q.E.D.

PROP. III. THEOR.

If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base shall have the same ratio which the other sides of the triangle have to one another: and if the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line, drawn from the vertex to the point of section, bisects the vertical angle.

Let the vertical angle BAC of any triangle ABC be bisected by the straight line AD, which cuts the base BC in D: BD shall be to DC as BA to AC.

Through C draw CE parallel to AD, meeting BA produced in E: Then, because AC meets the two parallels AD, EC, the angle DAC is equal to the alternate angle ACE: But the angle DAC is equal to the angle BAD; therefore also the angle BAD is equal to the angle ACE: Again, because BAE meets the two parallels AD, EC, the exterior angle BAD is equal to the interior and opposite angle AEC: But the angle BAD has been proved equal to the angle ACE; therefore also the angle ACE is equal to the angle AEC, and the side AC to the side AE: Now because AD is parallel to CE,



a side of the triangle BCE, therefore (6. 2) BD is to DC as BA to AE: But AE is equal to AC; therefore BD is to DC as BA to AC.

Next, let BD be to DC as BA to AC, and join AD: then the angle BAC shall be bisected by AD.

For, the same construction being made, because BD is to DC as BA to AC, and also BD is to DC as BA to AE (since AD is parallel to EC), therefore BA is to AC as BA to AE: Therefore (5. 9) AC is equal to AE and the angle ACE to the angle AEC: But the angle ACE is equal to the alternate angle DAC, and the angle AEC is equal to the exterior and opposite angle BAD; therefore the angle BAD is equal to the angle DAC, that is, the angle BAC is bisected by the straight line AD.

Wherefore, *If the vertical angle &c.* Q.E.D.

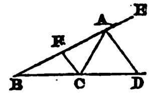
PROP. A. THEOR.

If the exterior angle of a triangle, made by producing any one of its sides, be bisected by a straight line which also cuts the base produced, the segments between the dividing line and the extremities of the base shall have the same ratio which the other sides of the triangle have to one another: and if the segments of the base produced have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section shall bisect the exterior angle of the triangle.

Let the exterior angle CAE of any triangle ABC be bisected by AD, which meets the base produced in D: then BD shall be to DC as BA to AC.

Through C draw CF parallel to AD: Then, because AC meets the two parallels AD, CF, the angle DAC is equal to the alternate angle ACF: But the angle

DAC is equal to the angle DAE; therefore also the angle DAE is equal to the angle ACF: Again, because FAE meets the two parallels AD, FC, the exterior angle DAE is equal to the interior and opposite angle AFC: But the angle DAE was proved equal to the angle ACF; therefore also the angle ACF is equal to the angle AFC, and the side AC to the side AF: Now because AD is parallel to CF, a side of the triangle BCF, therefore BD is to DC as BA to AF: But AF is equal to AC; therefore BD is to DC as BA to AC.



Next, let BD be to DC as BA to AC, and join AD: then the angle CAE shall be bisected by AD.

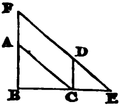
For, the same construction being made, because BD is to DC as BA to AC, and also BD is to DC as BA to AF, therefore BA is to AC as BA to AF, and therefore AC is equal to AF, and the angle ACF to the angle AFC: But the angle ACF is equal to the alternate angle CAD, and the angle AFC to the exterior angle DAE; therefore also the angle CAD is equal to the angle DAE, that is, the angle CAE is bisected by AD.

Wherefore, *If the exterior angle &c.* Q.E.D.

PROP. IV. THEOR.

The sides about the equal angles of equiangular triangles are proportionals, and those opposite to the equal angles are homologous sides, that is, are the antecedents or consequents of the ratios.

Let ABC, DCE, be equiangular triangles, having the angle ABC equal to the angle DCE, and the angle ACB to the angle DEC, and consequently the angle BAC equal to the angle CDE: then the sides about the equal angles of the triangles, ABC, DCE, shall be proportionals, those being the homologous sides which are opposite to the equal angles.



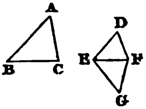
Let the triangle DCE be so placed that its side CE may be contiguous to BC, and in the same straight line with it: Then, because the angles ABC, ACB, are together less than two right angles, and that the angle ACB is equal to the angle DEC, therefore also the angles ABC, DEC are less than two right angles, and therefore BA, ED, if produced, shall meet; let them be produced and meet in F: Then, because the angle ABC is equal to the angle DCE, therefore BF is parallel to CD; and, again, because the angle ACB is equal to the angle DEC, therefore AC is parallel to FE: Therefore FACD is a parallelogram, and AF is equal to CD, and AC to FD: Now, because AC is parallel to FE, one of the sides of the triangle FBE, therefore BA is to AF as BC to CE: but AF is equal to CD; therefore BA is to CD as BC to CE, and, alternately (5. 16), AB is to BC as DC to CE: Again, because CD is parallel to BF, therefore BC is to CE as FD to DE: but FD is equal to AC; therefore BC is to CE as AC to DE, and, alternately, BC is to CA as CE to ED: Lastly, because it has been proved that AB is to BC as DC to CE, and BC to CA as CE to ED, therefore, ex æquali (5. 22), BA is to AC as CD to DE.

Wherefore, *The sides &c.* Q.E.D.

PROP. V. THEOR.

If the sides of two triangles, about each of their angles, be proportionals, the triangles shall be equiangular, and have those angles equal which are opposite to homologous sides.

Let ABC, DEF, be two triangles, which have their sides proportionals, so that AB is to BC as DE to EF, and BC to CA as EF to FD, and, consequently, ex æquali, BA to AC as ED to DF: then the triangles ABC, DEF, shall be equiangular, and shall have those angles equal which are opposite to the homologous sides, viz. the angle ABC to the angle DEF, and BCA to EFD, and BAC to EDF.



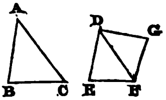
At the points E, F, in the straight line EF, make the angle FEG equal to the angle ABC, and the angle EFG to the angle BCA; therefore also the remaining angle EGF will be equal to the remaining angle BAC: Therefore the triangle ABC is equiangular to the triangle GEF, and, consequently (**6.** 4), they have their sides about the equal angles proportionals; therefore AB is to BC as GE to EF: But (*Hyp.*) AB is to BC as DE to EF; therefore (**5.** 11) DE is to EF as GE to EF, and therefore (**5.** 9) DE is equal to GE: And in like manner, DF is equal to GF: Therefore, because, in the two triangles DEF, GEF, the two sides DE, EF are equal to the two GE, EF, each to each, and the base DF to the base GF,—therefore the angle DEF is equal to the angle GEF, and the other angles to the other angles to which the equal sides are opposite, viz. the angle DFE to the angle GFE, and the angle EDF to the angle EGF: And because the angle DEF is equal to the angle GEF, and the angle GEF to the angle ABC, therefore also the angle ABC is equal to the angle DEF: And, for the like reason, the angle ACB is equal to the angle DFE, and the angle at A to the angle at D: Therefore the triangle ABC is equiangular to the triangle DEF.

Wherefore, *If the sides &c.* Q. E. D.

PROP. VI. THEOR.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.

Let ABC, DEF be two triangles, which have the angle BAC of the one equal to the angle EDF of the other, and the sides about those angles proportionals, that is, BA to AC as ED to DF: then the triangles ABC, DEF shall be equiangular, and shall have the angle ABC equal to the angle DEF, and the angle ACB to the angle DFE.



At the points D, F, in the straight line DF, make the angle FDG equal to either of the angles BAC, EDF, and the angle DFG equal to the angle ACB; therefore also the angle at B is equal to the angle at G, and the triangle ABC is equiangular to the triangle DGF: Therefore (6. 4) BA is to AC as GD to DF: But (*Hyp.*) BA is to AC as ED to DF; therefore GD is to DF as ED to DF, and therefore GD is equal to ED: Therefore, because in the two triangles GDF, EDF, the two sides GD, DF are equal to the two ED, DF, each to each, and the angle GDF to the angle EDF—therefore the base GF is equal to the base EF, and the triangle GDF to the triangle EDF, and the other angles to the other angles, each to each, to which the equal sides are opposite, viz. the angle DFG to the angle DFE, and the angle at G to the angle at E: But the angle DFG is equal to the angle ACB; therefore the angle ACB is equal to the angle DFE: And the angle BAC is equal to the angle EDF (*Hyp.*); therefore also the angle at B is equal to the angle at E, and the triangle ABC is equiangular to the triangle DEF.

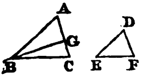
Wherefore, *If two triangles &c.* Q.E.D.

PROP. VII. THEOR.

If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals, then, if each of the remaining angles be either less, or not less, than a right angle, or if one of them be a right angle, the triangles shall be equiangular, and shall have those angles equal about which the sides are proportionals.

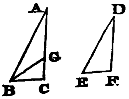
Let ABC, DEF be two triangles, which have one angle of the one equal to one angle of the other, viz. the angle BAC to the angle EDF, and the sides about two other angles ABC, DEF, proportionals, viz. AB to BC as DE to EF; and, first, let each of the remaining angles at C, F be less than a right angle: then the triangles ABC, DEF shall be equiangular, and shall have the angle ABC equal to the angle DEF, and the angle at C to the angle at F.

For, if the angles ABC, DEF be not equal, let ABC be the greater, and at the point B, in the straight line AB, make the angle ABG equal to the angle DEF: Then, because the angle BAG is equal to EDF, and ABG to DEF, therefore the angles AGB, DFE are equal, and the triangle ABG is equiangular to DEF: Therefore AB is to BG as DE to EF: But (*Hyp.*) AB is to BC as DE to EF; therefore AB is to BC as AB to BG, and therefore BC is equal to BG, and the angle BCG to the angle BGC: But BCG is less than a right angle; therefore also BGC is less than a right angle, and therefore AGB must be greater than a right angle: But the angle AGB was proved equal to DFE; therefore also the angle

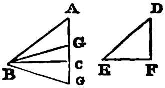


DFE is greater than a right angle: But it is also less (*Hyp.*)—which is absurd: Therefore the angle ABC is not unequal to DEF, that is, it is equal to it, and the angle at C to the angle at F.

Next, let each of the angles at C, F be not less than a right angle: Then, if the angle ABC be not equal to DEF, let ABC be the greater, and make the angle ABG equal to DEF: Then it may be proved, as before, that BC is equal to BG, and the angle BCG to the angle BGC: But BCG is not less than a right angle; therefore also BGC is not less than a right angle, that is, two angles of the triangle BCG are together not less than two right angles—which is impossible (I. 17): Therefore the angle ABC is equal to the angle DEF, and the angle at C to the angle at F.



Lastly, let one of the angles at C, F, viz. the angle at C, be a right angle: Then, if the angle ABC be not equal to the angle DEF, it must be either greater or less; in either case, make the angle ABG equal to DEF: Then, as before, it may be shewn that the angle BCG is equal to the angle BGC: But BCG is a right angle; therefore also BGC is a right angle, that is, two angles of the triangle BCG are together equal to two right angles—which is impossible: Therefore the angle ABC is equal to the angle DEF, and the angle at C to the angle at F.



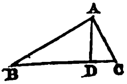
Wherefore, *If two triangles &c.* Q.E.D.

PROP. VIII. THEOR.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle, and to one another.

Let ABC be a right-angled triangle, having the right angle BAC; and from the point A, let AD be drawn perpendicular to the base BC: the triangles ABD, ACD shall be similar to the whole triangle ABC, and to one another.

Because the angle BAC is equal to the angle ADB, each of them being a right angle, and that the angle at B is common to the two triangles ABC, ABD, therefore the third angle ACB is equal to the third angle BAD: Therefore the triangles ABC, ABD are equiangular, and have the sides about their equal angles proportionals (6. 4), that is, they are similar (6. *Def.* 1): And, in like manner, it may be shewn that the triangles ABC, ACD are equiangular and similar: Therefore the triangles ABD, ACD, being both equiangular and similar to ABC, are equiangular and similar to each other.



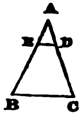
Wherefore, *In a right-angled triangle &c.* Q.E.D.

Cor. From this it is manifest, that the perpendicular, drawn from the right angle of a right-angled triangle to the base, is a mean proportional between the segments of the base, and also that each of the sides is a mean proportional between the base and its segment adjacent to that side: For, in the triangles ABD, ACD, BD is to DA as DA to DC, and in the triangles ABC, ABD, CB is to BA as BA to BD, and in the triangles ABC, ACD, BC is to CA as CA to CD.

PROP. IX. Prob.

From a given straight line to cut off any part required.

Let AB be the given straight line: it is required to cut off any part required from it (5. *Def.* 1).



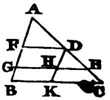
From the point A draw a straight line AC, making any angle with AB; and in AC take any point D, and take AC the same multiple of AD that AB is of the part which is to be cut off from it; join BC, and draw DE parallel to it: then AE is the part required.

For, because ED is parallel to BC, one of the sides of the triangle ABC, therefore BE is to EA as CD to DA, and, by composition (5. 18), BA is to AE as CA to AD: But CA is a certain multiple of AD; therefore (5. D), BA is the same multiple of AE, that is, whatever part AD is of AC, AE is the same part of AB: Wherefore, from the given straight line AB, the part required has been cut off. Q.E.F.

PROP. X. PROB.

To divide a given straight line similarly to a given divided straight line, that is, into parts that shall have the same ratios to one another, which the parts of the given divided line have.

Let AB be the straight line given to be divided, and AC the given divided line; it is required to divide AB similarly to AC.



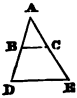
Let AC be divided in the points D, E; and let AB, AC be placed so as to contain any angle, and join BC; through the points D, E draw DF, EG parallel to BC, and through D draw DHK parallel to AB: Then each of the figures FH, HB is a parallelogram, and therefore DH is equal to FG, and HK to GB: Now, because HE is parallel to KC, one of the sides of the triangle DKC, therefore KH is to HD as CE to ED: But KH is equal to BG, and HD to GF; therefore BG is to GF as CE to ED: Again, because FD is parallel to GE, one of the sides of the triangle AGE, GF is to FA as ED to DA: And it has been proved that BG is to GF as CE to ED, and GF to FA as ED to DA; therefore the given straight line AB is divided similarly to AC. Q.E.F.

PROP. XI. PROB.

To find a third proportional to two given straight lines.

Let AB, AC be the two given straight lines, and let them be placed so as to contain any angle: it is required to find a third proportional to AB, AC.

Produce AB, AC, to the points D, E, and make BD equal to AC; join BC, and through D draw DE parallel to BC: Then, because BC is parallel to DE, a side of the triangle ADE, therefore AB is to BD as AC to CE: But BD is equal to AC; therefore AB is to AC as AC to CE: Wherefore, to the two given straight lines AB, AC, a third proportional CE has been found. Q.E.F.

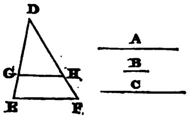


PROP. XII. PROB.

To find a fourth proportional to three given straight lines.

Let A, B, C, be three given straight lines: it is required to find a fourth proportional to A, B, C.

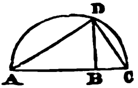
Take two straight lines, DE, DF, containing any angle EDF, and in these make DG equal to A, GE equal to B, and DH equal to C; join GH, and through E draw EF parallel to GH: Then, because GH is parallel to EF, one of the sides



of the triangle DEF, therefore DG is to GE as DH to HF: But DG is equal to A, GE to B, and DH to C; therefore A is to B as C to HF: Wherefore, to the three given straight lines A, B, C, a fourth proportional HF has been found. Q.E.F.

PROP. XIII. PROB.

To find a mean proportional between two given straight lines.



Let AB, BC be the two given straight lines: it is required to find a mean proportional between them.

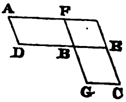
Place AB, BC in a straight line, and upon AC describe the semicircle ADC; from the point B draw BD at right angles to AC, and join AD, DC: Then the angle ADC in a semicircle is a right angle; and because in the right-angled triangle ADC, DB is drawn from the right angle perpendicular to the base, therefore (6. 8. *Cor.*) DB is a mean proportional between AB, BC, the segments of the base: Wherefore, between the two given straight lines AB, BC, a mean proportional DB has been found. Q.E.F.

PROP. XIV. THEOR.

Equal parallelograms, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and parallelograms, which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

First, let AB, BC be equal parallelograms, which have the angles at B equal: the sides of the parallelograms AB, BC about the equal angles shall be reciprocally proportional, that is, DB shall be to BE as GB to BF.

Let the parallelograms be placed, so that the sides DB, BE may be in the same straight line; then, since the angles DBF, EBF are together equal to two right angles, and that the angle DBF is equal to the angle GBE, therefore the angles GBE, EBF are together equal to two right angles, and therefore GB, BF are in the same straight line: Complete the parallelogram FE: Then, because the parallelogram AB is equal to BC, and that FE is another parallelogram, therefore AB is to FE as BC to FE: But as AB is to FE as the base DB to the base BE, and BC is to FE as the base GB to the base BF; therefore DB is to BE as GB to BF.



Next, let AB, BC be parallelograms, which have the angles at B equal and the sides about the equal angles reciprocally proportional, viz. DB to BE as GB to BF: the parallelogram AB shall be equal to the parallelogram BC.

For, the same construction being made, because DB is to BE as GB to BF, and that DB is to BE as the parallelogram AB to the parallelogram FE, and GB to BF as the parallelogram BC to the parallelogram FE, therefore AB is to FE as BC to FE, and therefore (5. 9) the parallelogram AB is equal to the parallelogram BC.

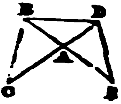
Wherefore, *Equal parallelograms &c.* Q.E.D.

PROP. XV. THEOR.

Equal triangles, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and triangles, which *have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.*

First, let ABC, ADE be equal triangles, which have the angle BAC equal to the angle DAE: the sides of the triangles about the equal angles shall be reciprocally proportional, that is, CA shall be to AD as EA to AB.

Let the triangles be placed so that their sides CA, AD may be in the same straight line, and therefore also EA and AB will be in the same straight line: join BD: Then, because the triangle ABC is equal to the triangle ADE, and that ABD is another triangle, therefore the triangle ABC is to the triangle ABD as the triangle ADE to the triangle ABD: But the triangle ABC is to the triangle ABD as the base CA to the base AD, and the triangle ADE is to the triangle ABD as the base EA to the base AB; therefore CA is to AD as EA to AB.



Next, let ABC, ADE be triangles, which have the angle BAC equal to the angle DAE, and the sides about the equal angles reciprocally proportional, viz. CA to AD as EA to AB: the triangle ABC shall be equal to the triangle ADE.

For, the same construction being made, because CA is to AD as EA to AB, and CA is to AD as the triangle ABC to the triangle ABD, and EA to AB as the triangle ADE to the triangle ABD, therefore the triangle ABC is to the triangle ABD as the triangle ADE to the triangle ABD, and therefore (5. 9) the triangle ABC is equal to the triangle ADE.

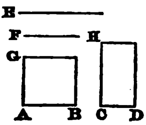
Wherefore, *Equal triangles &c.* Q. E. D.

PROP. XVI. THEOR.

If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means: and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.

First, let the four straight lines AB, CD, E, F, be proportionals, viz. let AB be to CD as E to F: the rectangle contained by AB and F shall be equal to the rectangle contained by CD and E.

From the points A, C, draw AG, CH at right angles to AB, CD; make AG equal to F, and CH equal to E; and complete the parallelograms BG, DH: Then, because AB is to CD as E to F, and that E is equal to CH and F to AG, therefore AB is to CD as CH to AG, that is, the sides of the parallelograms BG, DH about the equal angles, are reciprocally proportional, and therefore (6. 14) the parallelogram BG is equal to the parallelogram DH: But the parallelogram BG is contained by the straight lines AB and F, because AG is equal to F, and the parallelogram DH is contained by CD and E, because CH is equal to E; therefore the rectangle contained by AB and F is equal to the rectangle contained by CD and E.



Next, let the rectangle contained by AB and F be equal to the rectangle contained by CD and E: then these four lines shall be proportionals, viz. AB shall be to CD as E to F.

For, the same construction being made, because the rectangle AB, F, is equal to the rectangle CD, E, and that the rectangle BG is contained by AB and F, because

AG is equal to F, and the rectangle DH by CD and E, because CH is equal to E, therefore the parallelogram BG is equal to the parallelogram DH: And they are equiangular: But (6. 14) the sides about the equal angles of equal parallelograms are reciprocally proportional; therefore, AB is to CD as CH to AG: And CH is equal to E, and AG to F; therefore AB is to CD as E to F.

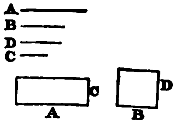
Wherefore, *If four straight lines &c.* Q. E. D.

PROP. XVII. THEOR.

If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean: and, if the rectangle contained by the extremes be equal to the square of the mean, the three straight lines are proportionals.

First, let the three straight lines A, B, C be proportionals, viz. A to B as B to C: the rectangle contained by A and C shall be equal to the square of B.

Take D equal to B: Then, because A is to B as B to C, and that B is equal to D, therefore A is to B as D to C: But, if four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means; therefore the rectangle A, C, is equal to the rectangle B, D: But the rectangle B, D is the square of B, because B is equal to D; therefore the rectangle A, C is equal to the square of B.



Next, let the rectangle contained by A, C be equal to the square of B: A shall be to B as B to C.

For, the same construction being made, because the rectangle A, C is equal to the square of B, and that the square of B is equal to the rectangle B, D, because B is equal to D, therefore the rectangle A, C is equal to the rectangle B, D: But (6. 16) if the rectangle contained by the extremes be equal to that contained by the means, the four straight lines are proportionals; therefore A is to B as D to C: But B is equal to D; therefore A is to B as B to C.

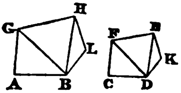
Wherefore, *If three straight lines &c.* Q.E.D.

PROP. XVIII. PROB.

Upon a given straight line to describe a rectilineal figure, similar and similarly situated to a given rectilineal figure.

Let AB be the given straight line, and CDEF the given rectilineal figure of four sides: it is required to describe upon the given straight line AB, a rectilineal figure, similar and similarly situated to CDEF.

Join DF, and at the points A, B, in the straight line AB, make the angle BAG equal to the angle DCF, and the angle ABG to the angle CDF; therefore also the remaining angle CFD is equal to the remaining angle AGB, and the triangle CFD is equiangular to the triangle AGB: Again, at the points B, G, in the straight line BG, make the angle GBH equal to the angle DFE, and the angle BGH to the angle DFE; therefore the remaining angle DEF is equal to the remaining angle BHG, and the triangle DEF is equiangular to the triangle BHG: Then, because the angle AGB is equal to the angle CFD, and the angle BGH to the angle DFE, therefore the whole angle AGH is equal to the



whole angle CFE: In like manner, the angle ABH is equal to the angle CDE: And the angle BAG is equal to the angle DCF, and the angle BHG to DEF: Therefore the rectilineal figure ABHG is equiangular to CDEF: Also these figures have their sides about the equal angles proportionals: For, because the triangles BAG, DCF are equiangular, therefore BA is to AG as DC to CF (6. 4): Again, for the like reason, AG is to GB as CF to FD, and GB to GH as FD to FE; therefore, ex æquali, AG is to GH as CF to FE: In like manner, it may be proved that AB is to BH as CD to DE: And GH is to HB as FE to ED: Therefore the figures ABHG, CDEF are equiangular, and have their sides about the equal angles proportionals: Wherefore (6. *Def.* 1) they are similar to one another.

Next, let it be required to describe, upon the given straight line AB, a rectilineal figure similar, and similarly situated, to the rectilineal figure CDKEF.

Join DE, and upon the given straight line AB describe, as in the former case, the rectilineal figure ABHG, similar, and similarly situated, to the quadrilateral figure CDEF; and at the points B, H, in the straight line BH, make the angle HBL equal to the angle EDK and the angle BHL equal to the angle DEK—therefore also the remaining angle at L will be equal to the remaining angle at K: Then, because the figures ABHG, CDEF are similar, the angle BHG is equal to the angle DEF: But the angle BHL is equal to the angle DEK; therefore the whole angle GHL is equal to the whole angle FEK: Similarly, the angle ABL is equal to the angle CDK: Therefore the five-sided figures ABLHG, CDKEF are equiangular: Also because the figures ABHG, CDEF are similar, therefore AB is to BH as CD to DE: But (6. 4) BH is to

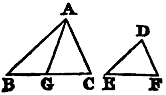
BL as DE to DK; therefore, ex equali, AB is to BL as CD to DK: In like manner, GH is to HL as FE to EK: and (6. 4) BL is to LH as DK to KE: Therefore the figures ABLHG, CDKEF are equiangular, and have their sides about the equal angles proportionals: Wherefore they are similar to one another: And, in like manner, upon a given straight line may be described a rectilineal figure, similar, and similarly situated, to a given rectilineal figure of six sides; &c. Q.E.F.

PROP. XIX. Theor.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let ABC, DEF be similar triangles, having the angle at B equal to the angle at E, and let AB be to BC as DE to EF, so that the side BC is homologous to EF: the triangle ABC shall be to the triangle DEF in the duplicate ratio of BC to EF.

Take BG a third proportional to BC, EF (6. 11), so that BC may be to EF as EF to BG, and join AG: Then, because AB is to BC as DE to EF, therefore, alternately (5. 16), AB is to DE as BC to EF: But BC is to EF as EF to BG; therefore AB is to DE as EF to BG, that is, the sides of the triangles ABG, DEF, about their equal angles, are reciprocally proportional, and therefore (6. 15) the triangle ABG is equal to the triangle DEF: And because BC is to EF as EF to BG, therefore (5. *Def.* 10) BC is to BG in the duplicate ratio of BC to EF: But, as BC to BG, so is the triangle ABC to the triangle ABG; therefore the triangle ABC is to the triangle ABG in the duplicate ratio of



BC to EF: But the triangle ABG is equal to tl
triangle DEF; therefore also the triangle ABC is to tl
triangle DEF in the duplicate ratio of BC to EF.

Wherefore, *Similar triangles &c.* Q.E.D.

COR. If three straight lines be proportionals, as tl
first is to the third, so is any triangle upon the first 1
a similar and similarly described triangle upon tl
second.

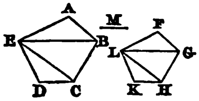
PROP. XX. THEOR.

Similar polygons may be divided into the same number
similar triangles, having the same ratio to one anoth
that the polygons have; and the polygons are to o
another in the duplicate ratio of their homologous sides.

Let ABCDE, FGHKL be similar polygons, and l
AB be the side homologous to FG: the polygo
ABCDE, FGHKL may be divided into the same nun
ber of similar triangles, whereof each to each has tl
same ratio which the polygons have to one anothe
and the polygon ABCDE is to the polygon FGHKL
the duplicate ratio of AB to FG.

Join BE, EC, GL, LH: Then, because the polyg
ABCDE is similar to the polygon FGHKL, the ang
BAE is equal to the angle GFL, and BA is to AE
GF to FL (6. *Def.* 1): Therefore the triangles AB
FGL have one angle of the one equal to one angle
the other, and the sides about the equal angles propo
tionals, and therefore they are equiangular (6. 6), a
therefore (6. 4) similar to one another; therefore tl
angle ABE is equal to the angle FGL: But, becau
the polygons are similar, the whole angle ABC is equ
to the whole angle FGH; therefore the remaining ang
EBC is equal to the remaining angle LGH: A

because the triangles ABE, FGL are similar, EB is to AB as LG to FG, and also, because the polygons are similar, AB is to BC as FG to GH; therefore, ex æquali, EB is to BC as LG to GH, that is, the triangles EBC, LGH, have the sides about their equal angles proportionals, and therefore they are equiangular and similar to one another: In like manner, it may be shewn that the triangle ECD is similar to the triangle LHK: Therefore the similar polygons ABCDE, FGHKL may be divided into the same number of similar triangles.



Also these triangles have, each to each, the same ratio which the polygons have to one another, the antecedents being ABE, EBC, ECD, and the consequents FGL, LGH, LHK: and the polygon ABCDE is to the polygon FGHKL in the duplicate ratio of AB to EG.

For, because the triangle ABE is similar to the triangle FGL, therefore (6. 19) ABE is to FGL in the duplicate ratio of EB to LG: For the like reason, the triangle EBC is to the triangle LGH in the duplicate ratio of EB to LG: Therefore the triangle ABE is to the triangle FGL as the triangle EBC to the triangle LGH: In like manner, the triangle EBC is to the triangle LGH as the triangle ECD to the triangle LHK: Therefore the triangle ABE is to the triangle FGL as the triangle EBC to the triangle LGH and as the triangle ECD to the triangle LHK: But (5. 12) as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents; therefore, as the triangle ABE to the triangle FGL, so is the polygon ABCDE to the polygon FGHKL: But the triangle ABE is to the triangle FGL in the duplicate ratio of AB to FG; therefore also the polygon ABCDE is to the polygon FGHKL in the duplicate ratio of AB to FG: And, in like manner, it may be proved, that similar figures of any number of sides more than three are to one another in the duplicate ratio of their homologous sides; and it has already been proved (6. 19) in the case of triangles.

Wherefore, universally, *Similar polygons &c.*

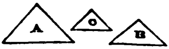
Cor. If to AB, FG, a third proportional M be taken, then (5. *Def.* 10) AB will be to M in the duplicate ratio of AB to FG: But any rectilineal figure upon AB will be to a similar and similarly described figure upon FG in the duplicate ratio of AB to FG: Therefore, as AB to M, so is the figure upon AB to the figure upon FG, as was before proved in the case of triangles (6. 19. *Cor.*): Therefore, universally, if three straight lines be proportionals, as the first is to the third, so is any rectilineal figure upon the first to a similar and similarly described rectilineal figure upon the second.

PROP. XXI. Theor.

Rectilineal figures, which are similar to the same rectilineal figure, are also similar to one another.

Let each of the rectilineal figures A, B, be similar to the rectilineal figure C: the figure A is similar to the figure B.

Because A is similar to C, they are equiangular, and have their sides about the equal angles proportionals (6. *Def.* 1): Again, because B is similar to C, they are equiangular, and have their



sides about the equal angles proportionals: Therefore the figures A, B are each of them equiangular to C, and have the sides about the equal angles of each of them, and of C, proportionals: Therefore the figures A and B are equiangular, and have their sides about the equal angles proportionals; and therefore the figure A is similar to the figure B.

Wherefore, *Rectilineal figures &c.* Q.E.D.

PROP. XXII. THEOR.

If four straight lines be proportionals, and similar rectilineal figures be similarly described upon the first and second, and also upon the third and fourth, these shall also be proportionals: and if the similar rectilineal figures similarly described upon the first and second, and also upon the third and fourth, of four straight lines be proportionals, those straight lines shall also be proportionals.

First, let the four straight lines AB, CD, EF, GH be proportionals, viz. AB to CD as EF to GH; and upon AB, CD let the similar rectilineal figures KAB, LCD be similarly described, and upon EF, GH the similar rectilineal figures MF, NH: the figure KAB shall be to the figure LCD as the figure MF to the figure NH.

To AB, CD, take a third proportional X, and to EF, GH, a third proportional O: Then, because AB is to CD as EF to GH, and that AB is to CD as CD to X, and EF to GH as GH to O, therefore (5. 11) CD is to X as GH to O: And AB is to CD as EF to GH; therefore, ex æquali, AB is to X as EF to O: But, as AB to X, so is the figure KAB to the figure LCD (6. 20. *Cor.*), and as EF to O, so is the

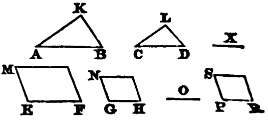


figure MF to the figure NH; therefore the figure KAB is to the figure LCD as the figure MF to the figure NH.

Next, let the figure KAB be to the figure LCD as the figure MF to the figure NH: then AB shall be to CD as EF to GH.

Make (6. 12), as AB to CD, so EF to PR; and upon PR (6. 18) describe the rectilineal figure SR, similar and similarly situated to either of the figures MF, NH: Then, because AB is to CD as EF to PR, and that upon AB, CD are described the similar and similarly situated figures KAB, LCD, and upon EF, PR, the similar and similarly situated figures MF, SR, therefore KAB is to LCD as MF to SR: But (*Hyp.*) KAB is to LCD as MF to NH; therefore MF is to SR as MF to NH, and therefore the figures SR, NH, are equal: But they are also similar, and similarly situated; therefore PR is equal to GH: And, because AB is to CD as EF to PR, and that PR is equal to GH, therefore AB is to CD as EF to GH.

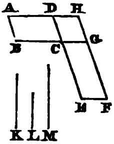
Wherefore, *If four straight lines &c.* Q. E. D.

PROP. XXIII. THEOR.

Equiangular parallelograms are to one another in the ratio which is compounded of the ratios of their sides.

Let AC, CF be equiangular parallelograms, having the angle BCD equal to the angle ECG: the parallelogram AC shall be to the parallelogram CF in the ratio which is compounded of the ratios of their sides.

Let BC, CG be placed in a straight line; therefore DC and CE will be also in a straight line: complete the parallelogram DG, and, taking any straight line K, make (6. 12) K to L as BC to CG, and L to M as DC to CE; then the ratios of K to L and L to M are the same as the ratios of the sides, viz. of BC to CG and DC to CE: But the ratio of K to M (5. *Def.* A) is that which is said to be compounded of the ratios of K to L and L to M; therefore also the ratio of K to M is the ratio compounded of the ratios of the sides: Now the parallelogram AC is to the parallelogram CH as BC to CG; but BC is to CG as K to L; therefore also the parallelogram AC is to the parallelogram CH as K to L: Again, the parallelogram CH is to the parallelogram CF as DC to CE; but DC is to CE as L to M; therefore also the parallelogram CH is to the parallelogram CF as L to M: Since, then, it has been proved that the parallelogram AC is the parallelogram CH as K to L, and the parallelogram CH to the parallelogram CF as L to M, therefore, ex æquali, the parallelogram AC is to the parallelogram CF as K to M: But K is to M in the ratio which is compounded of the ratios of the sides; therefore also the parallelogram AC is to the parallelogram CF in the ratio which is compounded of the ratios of the sides.



Wherefore, *Equiangular parallelograms &c.* Q.E.D.

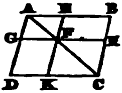
PROP. XXIV. THEOR.

The parallelograms, which are about the diameter of any parallelogram, are similar to the whole and to one another.

Let ABCD be a parallelogram, of which the diameter is AC, and EG, HK the parallelograms about the diameter: the parallelograms EG, HK shall be similar to the whole parallelogram ABCD, and to one another.

For, because DC, GF are parallels, the angle ADC is equal to the angle AGF: For the like reason, because

BC, EF are parallels, the angle ABC is equal to the angle AEF: And each of the angles BCD, EFG is equal to the opposite angle BAD, and therefore to one another: Therefore the parallelograms ABCD, AEFG are equiangular: And because the angle ABC is equal to the angle AEF, and the angle BAC common to the two triangles BAC, EAF, therefore they are equiangular and similar to one another, and AB is to BC as AE to EF: But the opposite sides of parallelograms are equal to one another; therefore AB is to AD as AE to AG, and CD to CB as FG FE, and CD to DA as FG to GA: Therefore the parallelograms ABCD, AEFG have the sides about their equal angles proportionals, and are therefore similar to one another: For the same reason the parallelograms ABCD, FHCK are similar to one another: Therefore each of the parallelograms EG, HK is similar to BD, and therefore (6. 21) the parallelogram EG is similar to the parallelogram KH.



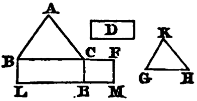
Wherefore, *The parallelograms &c.* Q.E.D.

PROP. XXV. PROB.

To describe a rectilineal figure which shall be similar to one given rectilineal figure and equal to another.

Let ABC be the given rectilineal figure to which the figure to be described is to be similar, and D that to which it is to be equal: it is required to describe a rectilineal figure, similar to ABC and equal to D.

Upon BC describe the parallelogram BE equal to the figure ABC, and upon CE describe the parallelogram CM equal to D, and having the angle FCE equal to the angle CBL; therefore BC and CF will be in the same straight line, as also LE and EM: between BC and CF find a mean proportional GH (6. 13), and upon GH describe the rectilineal figure KGH, similar and similarly situated to the figure ABC (6. 18): Then, because BC is to GH as GH to CF, and that, if three straight lines be proportionals, as the first is to the third, so is any figure upon the first to a similar and similarly described figure upon the second, therefore, as BC to CF, so is the figure ABC to the figure KGH: But, as BC to CF, so is the parallelogram BE to the parallelogram EF; therefore the figure ABC is to the figure KGH as the parallelogram BE to the parallelogram EF: And the figure ABC is equal to the parallelogram BE; therefore (5. 14) the figure KGH is equal to the parallelogram EF: But the parallelogram EF is equal to the figure D; therefore also the figure KGH is equal to the figure D, and it is similar to ABC: Therefore the rectilineal figure KGH has been described, similar to the figure ABC and equal to D. Q.E.F.

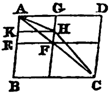


PROP. XXVI. THEOR.

If two similar parallelograms have a common angle and be similarly situated, they are about the same diameter.

Let the parallelograms ABCD, AEFG be similar and similarly situated, and have the common angle BAD: ABCD and AEFG shall be about the same diameter.

For, if not, let, if possible, the parallelogram BD have its diameter AHC in a different straight line from AF, the diameter of the parallelogram EG; let GF meet AHC in H, and through H draw HK parallel to AD or BC: Then the parallelograms ABCD,



AKHG, being about the same diameter, are s
one another (6. 24); therefore DA is to AB a
AK: But, because ABCD and AEFG ar
parallelograms, DA is to AB as GA to AE: T
GA is to AK as GA to AE, and, consequentl
AK is equal to AE, the less to the greate
is impossible: Therefore the parallelograms
AEFG cannot but have their diameters in t
straight line, that is, they are about the same di

Wherefore, *If two similar &c.* Q.E.D.

PROP. XXVII.—XXIX.

PROP. XXX. PROB.

To cut a given straight line in extreme and mean

Let AB be the given straight line: it is req
cut it in extreme and mean ratio.

Divide AB in the point C, so that the rectan
tained by AB, BC, may be equal to the square
of AC (2. 11): Then, because the rectangle
AB, BC is equal to the square of AC, therefore
AB is to AC as AC to CB: therefore (6. *Def.*
is cut in extreme and mean ratio in C. Q.E.F.

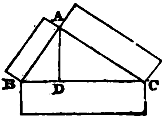
PROP. XXXI. THEOR.

*In any right-angled triangle, any rectilineal figure a
upon the side subtending the right angle is equa
similar and similarly situated figures upon the si
taining the right angle.*

Let ABC be a right-angled triangle, having th
angle BAC, and upon AB, AC, BC let any simi

similarly situated figures be described: the figure upon BC shall be equal to the similar and similarly situated figures upon BA, AC.

Draw the perpendicular AD: Then, because in the right-angled triangle ABC, AD is drawn, from the right angle at A, perpendicular to the base BC, therefore (6. 8) the triangles ADB, ADC are similar to the whole triangle ABC, and to one another: And because the triangle ABC is similar to the triangle ABD, therefore CB is to BA as BA to BD: But when three straight lines are proportionals, as the first is to the third, so is any figure upon the first to the similar and similarly described figure upon the second (6. 20. *Cor.*); therefore as CB to BD, so is the figure upon CB to the similar and similarly situated figure upon BA, and inversely, as BD to BC, so is the figure upon BA to the figure upon BC: In like manner, as CD to CB, so is the figure upon CA to the figure upon CB: Therefore (5. 24) as BD and CD together to BC, so are the figures upon BA, AC together to the figure upon BC: But BD and CD together are equal to BC; therefore (5. A.) the figure upon BC is equal to the similar and similarly situated figures on BA, AC.

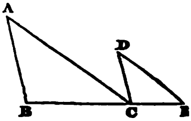


Wherefore, *In any right-angled triangle &c.* Q. E. D.

PROP. XXXII. THEOR.

If two triangles, which have two sides of the one proportional to two sides of the other, be joined at one angle, so as to have their homologous sides parallel to one another, the remaining sides shall be in a straight line.

Let ABC, DCE be two triangles which have the two sides BA, AC proportional to the two CD, DE, viz. BA to AC as CD to DE, and let AB be parallel to DC, and AC to DE: then BC and CE shall be in a straight line.



For, because AB is parallel to DC, and AC meets them, the alternate angles BAC, ACD are equal: but, for the like reason, the angle ACD is equal to the angle CDE; therefore also the angle BAC is equal to the angle CDE: And because the triangles ABC, DCE have the angle at A equal to the angle at D, and the sides about the equal angles proportionals, therefore (6. 6) the triangle ABC is equiangular to the triangle DCE, and the angle ABC is equal to the angle DCE: But the angle BAC was proved equal to the angle ACD; therefore the angles ABC, BAC are together equal to the angle ACE: To each of these equals add the angle ACB; then the angles ABC, BAC, ACB are together equal to the angles ACB, ACE: But ABC, BAC, ACB are together equal to two right angles; therefore also the angles ACB, ACE are together equal to two right angles, and therefore BC and CE are in a straight line.

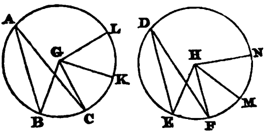
Wherefore, *If two triangles &c.* Q. E. D.

PROP. XXXIII. THEOR.

In equal circles, angles, whether at the centres or circumferences, have the same ratio which the circumferences on which they stand have to one another: and so also have the sectors.

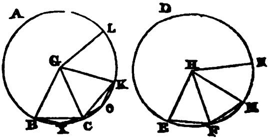
Let ABC, DEF be equal circles, and let BGC, EHF be angles at their centres, and BAC, EDF angles at their circumferences: as the circumference BC to the circumference EF, so shall the angle BGC be to the angle EHF, and the angle BAC to the angle EDF; and so also shall the sector BGC be to the sector EHF.

Take any number of circumferences CK, KL, each equal to BC, and any number whatever FM, MN, each equal to EF; and join GK, GL, HM, HN: Then, because the circumferences BC, CK, KL are all equal, the angles BGC, CGK, KGL are also equal; and therefore, whatever multiple the circumference BL is of the circumference BC, the same is the angle BGL of the angle BGC: In like manner, whatever multiple the circumference EN is of the circumference EF, the same is the angle EHN of the angle EHF: Now, if the circumference BL be equal to the circumference EN, the angle BGL is also equal to the angle EHN, and if greater, greater, and if less, less: But the circumference BL and the angle BGL are any equimultiples of the circumference BC and angle BGC, and the circumference EN and the angle EHN are any equimultiples of the circumference EF and the angle EHF; therefore (5. *Def.* 5) as the circumference BC is to the circumference EF, so is the angle BGC to the angle EHF: But as the angle BGC is to the angle EHF, so is the angle BAC to the angle EDF (5. 15), for each is double of each; therefore, as the circumference BC is to the circumference EF, so is the angle BGC to the angle EHF, and the angle BAC to the angle EDF.



Also, as the circumference BC is to the circumference EF, so shall the sector BGC be to the sector EHF.

Join BC, CK, and in the circumferences BC, CK take any points X, O, and join BX, XC, CO, OK: Then, because in the triangles BGC, CGK, the two sides BG, GC are equal to the two CG, GK, each to each, and that they contain equal angles—therefore the base BC is equal to the base CK, and the triangle BGC to the triangle CGK: And, because the circumference BC is equal to the circumference CK, the remaining part of the whole circumference, when BC is taken from it, is equal to the remaining part of the same circumference, when CK is taken from it: Therefore (3. 27) the angle BXC is equal to the angle COK, and the segment BXC is therefore similar to the segment COK: But they are upon equal straight lines BC, CK; therefore the segment BXC is equal to the segment COK: And the triangle BGC is equal to the triangle CGK; therefore the whole, the sector BGC, is equal to the whole, the sector CGK: In like manner, the sector KGL is equal to each of the sectors BGC, CGK, and so also the sectors EHF, FHM, MHN may be proved equal to one another: Therefore, whatever multiple the circumference BL is of the circumference BC, the same is the sector BGL of the sector BGC; and whatever multiple the circumference EN is of the circumference EF, the same is the sector EHN of the sector EHF: Now, if the circumference BL be equal to the circumference EN, the sector BGL will be equal to the sector EHN, and if greater, greater, and if less, less: But the circumference BL and the sector BGL are any equimultiples of the circumference BC and the sector BGC, and the circum-



ference EN and the sector EHN are any equimultiples of the circumference EF and the sector EHF; therefore, as the circumference BC is to the circumference EF, so is the sector BGC to the sector EHF.

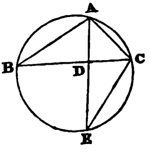
Wherefore, *In equal circles &c.* Q. E. D.

PROP. B. THEOR.

If the vertical angle of a triangle be bisected by a straight line which likewise cuts the base, the rectangle contained by the sides of the triangle shall be equal to the rectangle contained by the segments of the base, together with the square of the straight line bisecting the angle.

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD, which cuts the base BC in D: the rectangle BA, AC shall be equal to the rectangle BD, DC, together with the square of AD.

About the triangle describe the circle ABC, and produce AD to the circumference in E, and join EC: Then, because the angle BAD is equal to the angle CAE, and the angle ABD to the angle AEC, (being angles in the same segment of a circle,) the triangles ABD, AEC are equiangular to one another: Therefore BA is to AD as EA to AC; and therefore (6. 16) the rectangle BA, AC is equal to the rectangle EA, AD, that is (2. 3) to the rectangle ED, DA, together with the square of AD: But the rectangle ED, DA is equal to the rectangle BD, DC (3. 35); therefore the rectangle BA, AC is equal to the rectangle BD, DC, together with the square of AD.

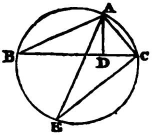


Wherefore, *If the vertical angle &c.* Q. E. D.

PROP. C. Theor.

If from any angle of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

Let ABC be a triangle, and let AD be the perpendicular from the angle A to the base BC: the rectangle BA, AC shall be equal to the rectangle contained by AD and the diameter of the circle described about the triangle.



About the triangle describe the circle ABC, and draw the diameter AE, and join EC: Then, because the right angle BDA is equal to the angle ECA in a semicircle, and the angle ABD to the angle AEC in the same segment, therefore the triangles ABD, AEC are equiangular: Therefore BA is to AD as EA to AC, and therefore the rectangle BA, AC is equal to the rectangle EA, AD.

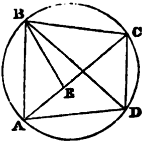
Wherefore, *If from any angle &c.* Q.E.D.

PROP. D. Theor.

The rectangle, contained by the diagonals of a quadrilateral inscribed in a circle, is equal to the sum of the rectangles contained by its opposite sides.

Let ABCD be any quadrilateral inscribed in a circle, and join AC, BD: the rectangle contained by AC, BD is equal to the two rectangles, contained by AB, CD, and by AD, BC.

Make the angle ABE equal to the angle DBC: Then, because the angle ABE is equal to the angle DBC, and the angle BAE to the angle BDC in the same segment, the triangle ABE is equiangular to the triangle DBC: Therefore BA is to AE as BD to DC, and therefore the rectangle AB, CD is equal to the rectangle BD, AE: Again, because the angle ABE is equal to the angle DBC, to each of these add the angle EBD; therefore the angle ABD is equal to the angle EBC: And the angle BDA is equal to the angle BCE in the same segment; therefore the triangle ABD is equiangular to the triangle EBC: Therefore AD is to DB as EC to CB, and therefore the rectangle AD, BC is equal to the rectangle BD, EC: But the rectangle AB, CD has been proved equal to the rectangle BD, AE; therefore (**2.** 1) the whole rectangle BD, AC is equal to the two rectangles AB, CD and AD, BC.



Wherefore, *The rectangle &c.* Q.E.D.

PROBLEMS.

BOOK VI.

1. The rectangle contained by two lines is a mean proportional between their squares.

2. A quadrilateral is such that the perpendiculars upon a diagonal from opposite angles are equal: divide it into four equal triangles by lines drawn to the angles from a point within it.

3. From one angle of a triangle a line is drawn to the bisection of the opposite side, and through its middle point another is drawn from either angle to the side subtending it: prove that this side is divided in the ratio of 2 to 1.

4. Through a given point draw a line so that, if perpendiculars be dropped upon it from two other given points, the parts intercepted between the given point and the feet of the perpendiculars may be equal.

5. If any three equal lines be drawn from the angles of a triangle to the opposite sides or these produced, and from any point within it lines parallel to these be drawn to the sides, the sum of these latter lines shall be equal to either of the former.

6. If three circles touch each other, two of which are equal, the vertical angle of the triangle, formed by joining the points of contact, is equal to the angle at the base of the isosceles triangle, formed by joining the three centres.

7. ABC is an equilateral triangle, E any point in AC; in BC produced take CD, CF equal to CA, CE, respectively, and let AF, DE, intersect in H: shew that

$$HC : EC :: AC : AC + EC.$$

8. The diagonals, AC, BD, of an inscribed quadrilateral meet in E: shew that AB.BC : AD.DC :: BE : ED.

9. If a square be inscribed in a right-angled triangle, one side coinciding with the hypothenuse, the base is divided in continued proportion.

10. If ABC be an inscribed triangle, and BD be drawn parallel to the tangent at A to meet AC, or AC produced, shew that AB is a mean proportional between AC, AD.

11. AB is divided in C, D, so that AB : AC :: AC : AD; if AE be any other line taken equal to AC, shew that the angle BED is bisected by EC.

12. The part of a tangent to a circle, intercepted by tangents at the extremities of any diameter, is divided at the point of contact so that the radius is a mean proportional between the two segments.

13. If two chords in a circle intersect so that the segments of one have the same ratio as those of the other, the line bisecting the angle between homologous segments will pass through the centre of the circle.

14. If a triangle be inscribed in a semicircle, and a perpendicular be drawn from any point in the diameter, meeting the circumference and the other two sides, the three segments of it thus made will be in continued proportion.

15. If the diagonals of an inscribed quadrilateral intersect at right angles, the rectangles of the opposite sides are together double of the area of the figure.

16. If chords, drawn from any fixed point in the circumference of a circle, be cut by another chord, parallel to the tangent at that point, the rectangle of each chord and the part of it between the given point and given chord is constant.

17. If in similar triangles, from any two equal angles, lines be drawn to the opposite sides, making equal angles with homologous sides, these will have the same ratio as the sides on which they fall, and will also divide them proportionally.

18. Apply Prop. II. to shew how a line may be drawn through a given point, parallel to a given line, by means of a piece of string.

19. In any triangle, right-angled at A, if CD be drawn bisecting the angle C, shew that

$$AB : AC :: BC - AC : AD.$$

20. If two circles touch externally, the part of their common tangent between the points of contact is a mean proportional between the diameters.

21. Given the lengths of three lines, drawn from the angles to the points of bisection of the opposite sides, to construct the triangle.

22. Inscribe a square in a given segment of a circle.

23. Find the locus of points dividing in a given ratio lines drawn from a given point to meet a given circle.

24. Given two circles which intersect, draw through either point of intersection a line cutting the circles, so that the chords intercepted may be in a given ratio.

25. Inscribe in a given triangle a parallelogram similar to a given parallelogram.

26. If through the vertex and extremities of the base of a triangle any two circles be described so as to intersect in the base or base produced, their diameters will be proportional to the respective sides.

27. If *any* two lines which intersect are cut by two given lines, the ratio of the rectangles of their respective segments will be the same as that for any other two lines, drawn parallel to them, and cut by the same given lines.

28. A, B, are the centres of two unequal circles, AP, BQ, *any* pair of parallel radii: shew that PQ passes through a fixed point, whose distances from the centres are proportional to the respective radii; and hence deduce a method of drawing a common tangent to two circles.

29. If through the fixed point in [28] *any* line be drawn cutting the circles, the radii at the points of section will be parallel, and the intercepted chords proportional to them.

30. If through the fixed point in [28] any two lines be drawn cutting the circles, the chords joining corresponding points of section will be parallel; and, of the eight points of section, any four lie in the circumference of a circle, provided that no two of them be homologous points, nor at once in the same circle and line.

31. CDE is a common tangent to two circles, meeting the line of centres, AB, in E; FGHKE is any line cutting the circles: shew that

$$EC.ED = EF.EK = EG.EH.$$

32. If *any* circle be drawn touching two other given circles, the lines joining the points of contact will pass through a fixed point.

33. Find the Arithmetic, Geometric, and Harmonic means between three given lines.*

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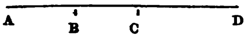
* Three straight lines AB, AC, AD, are said to be
(i) in *Arith. Prog.*, when $AB : AB :: AB \sim AC : AC \sim AD$,
(ii) in *Geom. Prog.*, when $AB : AC :: AB \sim AC : AC \sim AD$,
(iii) in *Harm. Prog.* when $AB : AD :: AB \sim AC : AC \sim AD$.
Hence it will be found that if AP, AQ, AR, be, respectively, the Ar., Geom., and Harm. means between AB and AC,

$$AP = \tfrac{1}{2}(AB + AC), \quad AQ^2 = AB.AC, \quad AR = \frac{2AB.AC}{AB+AC},$$

whence also $AP.AR = AQ^2$, or the three means are in continued proportion.

The Proportion in (iii) is often thus expressed,
$AB{:}AD{::}CB{:}CD$, or $AD{:}AB{::}CD{:}CB$ or $AD.BC=AB.CD$.



34. If the two angles, which one straight line ma with another on the same side of it, be bisected, sh that any line cutting four of the lines thus drawn, be harmonically divided; and, conversely, if a line divided harmonically, and from any point lines drawn to the four points of section, of which any t alternate ones contain a right angle, then the an between the other two will be bisected.

35. If from any point P without a given circle t gents PC, PD be drawn to it, and the line, CD, join the points of contact, cut in Q the diameter AO which passes through P, shew that OP.OQ=(rad. and that the line PB is harmonically divided, and t angle PCQ bisected.*

36. If in [35] *any* line be drawn through P cutt the circle in E, G, and CD in F, shew that PG is h monically divided, and the angles PEQ, PGQ bisect by EA, GA.

37. If P be the pole of CQD, then the pole of a line through P will lie in CQD: and, conversely, t polar of any point in CQD will pass through P.

38. Apply [37] to shew that, if a circle inscribed a quadrilateral, ABCD, touch its sides AB, &c. E, F, G, H, and EH, FG, be produced to meet in then KO drawn to the centre is perpendicular to AC.

39. In the triangle ABC, AC=2BC: if CD, C bisect the angle C and the exterior angle formed producing AC, then the triangles CBD, ACD, AB CDE, have their areas as 1, 2, 3, 4.

40. The locus of the vertices of all triangles, up the same base and having their sides in a given rat is a circle.

41. Find the locus of points at which two giv circles will be seen under the same angle.

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* The point P is called the *pole* of CQD, and CQD the *polar* of

42. In any triangle, ABC, right-angled at A, bisect the angle C by CD, and shew that

$$2AC^2 : AC^2 - AD^2 :: AB : AD.$$

43. Construct a triangle, having given one side, the angle opposite to it, and the ratio of the other two sides.

44. If two triangles have one angle of the one equal to one angle of the other and another angle of the one supplementary to another angle of the other, the sides opposite to these four angles will be proportionals.

45. Through any point in the line bisecting an angle, draw a line cutting the sides at equal angles: this is the shortest line that can be drawn through the given point to cut them, and the triangle so cut off is the least possible.

46. The line, which cuts at equal angles the lines containing a given angle, is the least that can be drawn to cut off a triangle of given area.

47. If an isosceles triangle be inscribed in a circle, and from the vertical angle a line be drawn to meet the circumference and base, the rectangle of the segments of this line is equal to the square of either of the sides of the triangle.

48. Within a given circle place six equal circles touching one another and the given circle; and shew that the interior circle, which touches them all, is equal to each of them.

49. The perpendiculars from the angles of a triangle upon the opposite sides will meet in a point.

50. AB is the diameter of a circle, AC, BD, are any two chords intersecting in E: draw EF perpendicular to AB, and shew that, if produced, it passes through the intersection of AD, BC.

51. Find a point without a given circle, such that the sum of the two lines, drawn from it touching the circle, shall be equal to the line drawn from it through the centre to meet the circle.

52. ABC is an isosceles triangle; draw AD perpendicular to the base, and DEF, cutting AB, AC, in E, F: then AD : DE :: AB + AF : AB − AF.

53. From any point P, tangents PA, PB, are drawn to a circle, and AC is drawn perpendicular to the diameter BD: shew that AC is bisected by PD in E.

54. Divide a given line into any number of equal parts; and thence shew how to divide a triangle into the same number of equal parts, by lines drawn from a point in one of its sides.

55. AD is drawn bisecting the vertical angle of a triangle, and cutting the base BC in D; in BC produced take a point E, equally distant from A and D, and shew that BE : DE :: DE : CE.

56. If through the bisection of the base of a triangle any line be drawn, cutting one side of the triangle, the other produced, and a line drawn parallel to the base from the vertex, this line shall be cut harmonically.

57. If four diverging lines cut a straight line harmonically, they will cut any other intercepted line harmonically.

58. Let the two circles, radii R and r, which touch (i) the three sides of a triangle ABC, and (ii) one side BC and the other sides produced, touch AB in D_1, D_2, AC in E_1, E_2: then shew that $BD_1 . BD_2 = CE_1 . CE_2 = Rr$.

59. AD is drawn perpendicular on the hypothenuse BC of a right-angled triangle: if R be the radius of the circle inscribed in ABC, and r, r', of those in ABD, ACD, shew that $R^2 = r^2 + r'^2$.

60. If the exterior angle of a triangle be bisected, by a line which cuts the base produced, the square of this line will be equal to the difference of the rectangles of the segments of the base and of the sides of the triangle.

61. If from the angle A of any parallelogram any line be drawn cutting the diagonal in E, and the sides BC, CD, in F, G, shew that AE is a mean proportional between EF, EG.

62. Given the nth part of a given line, find the $(n+1)^{\text{th}}$ part.

63. CAB, CEB, are two triangles which have a common angle B, and the sides CA, CE, equal: if, in BE produced, there be taken ED, a third proportional to BA, AC, then will the triangles BDC, BAC, be similar.

64. APB is the quadrant of a circle, SPT a tangent at P, cutting the radii OA, OB, in S, T: draw PM perpendicular to OA, and shew that the triangle

$$SOT : AOB :: ACB : OMP.$$

65. Through a given point draw a line, which, if produced, would pass through the point of intersection of two given lines, without producing them to meet.

66. If two triangles are equal, and have the sides about one angle in each reciprocally proportional, these angles are either equal or supplementary to each other.

67. Construct an isosceles triangle equal to a given scalene triangle, and with the same vertical angle.

68. ABCD is a rectangle; draw any line AE to CD and BF perpendicular to AE, and shew that the rectangle AE, BF, is equal to the given rectangle, ABCD.

69. ABC is an inscribed triangle, AD, AE, lines drawn to the base, parallel to the tangents at B, C: shew that $AD = AE$, and $BD : CE :: AB^2 : AC^2$.

70. In any triangle, if a perpendicular be dropped from the vertex on the base, the base : sum of sides :: diff. of sides : diff. or sum of segments of the base, according as the perpendicular falls within or without the triangle.

71. If any line ABCDE be drawn cutting two intersecting circles, C being the point in which it meets their line of section, then

$$AB : BC :: ED : DC, \text{ and } AE^2 : BD^2 :: AC.CE : BC.CD.$$

72. If a rectangle be inscribed in a right-angled triangle, having the right-angle common, the rectangle of the segments of the hypothenuse will be equal to the sum of the rectangles of the segments of the sides.

73. If an isosceles triangle be inscribed in a circle, having each of the sides double of the base, shew that the square upon the radius : that upon one of the sides :: 4 : 15.

74. ABC is a triangle, right-angled at A, and having the angle B double of the angle C; draw BD bisecting the angle B, and AE, DF, perpendiculars on BC, and shew that

$$\frac{1}{BE.DF} - \frac{1}{AE.BF} = \frac{1}{AE.BE}.$$

75. If a line touch a circle, and a perpendicular be drawn from the point of contact on any diameter, and if from the extremities of this diameter and from the centre perpendiculars to the diameter be drawn to meet the tangent, the four perpendiculars will be proportionals.

76. Inscribe a square in a given regular pentagon.

77. Let the lines AB, AC, be cut proportionally in D, E, and let perpendiculars at D, E, intersect in F: then shew that, for all such positions of D and E, F will lie in a fixed line through A.

78. AB is any chord of a circle; AC, BC, are drawn to any point in the circumference, and cut the diameter perpendicular to AB in D, E; if O be the centre, shew that $OD.OE = (\text{rad})^2$.

79. If semicircles be described on the segments of the hypothenuse made by a perpendicular from the vertex of a right-angled triangle, the segments of the sides intercepted by them will be in the triplicate ratio of the sides.

80. The sides containing a given angle are in a given ratio, and the vertex is fixed: shew if the extremity of one of the sides moves in a given line, so also does the extremity of the other.

81. If from the extremities of the base of a triangle two lines be drawn, each parallel to one of the sides and equal to the other, the lines joining their other

extremities with the other extremities of the base will cut off equal segments from the sides, each of them a mean proportional between the other two segments.

82. AB, AC are the sides of a regular pentagon and decagon inscribed in a circle whose centre is O; if OD be drawn to AB, bisecting the angle AOC, shew that the triangles ACB, ACD, as also the triangles AOB, DOB, are similar, and hence that

$$AB^2 - AC^2 = (\text{rad})^2.$$

83. Two circles touch in C; if any point D be taken without them, such that the radii AC, BC subtend equal angles at D, and DE, DF be tangents to the circles, then $DE.DF = DC^2$.

84. Bisect a triangle by a line drawn parallel to one of its sides.

85. One side of a right-angled triangle is double of the other: shew that the perpendicular from vertex on hypothenuse will cut off parts in the ratio of 1 : 4.

86. If two triangles are to one another as their bases, they have the same altitude; and triangles and parallelograms of unequal altitudes are to one another in the ratio compounded of the ratios of their bases and altitudes.

87. Describe a rhombus, equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

88. If two triangles have one angle of the one *equal* or *supplementary* to one angle of the other, they are to one another in the ratio compounded of the ratios of the sides containing them.

89. Describe a square, having given the difference between the diagonal and a side.

90. ABCD is a quadrilateral; if any line be drawn cutting AB, CD, in a, d, AD, BC, in b, c, and AC, BD, in e, f, then $ab : cd :: af \times be : cf \times de$.

91. Prove that the ratio of the diagonal of a square to the side is incommensurable, and thence shew that its numerical value is nearly 1.4142.

92. Through a given point describe a circle touching a given line and a given circle.

93. Describe a circle touching two given circles and a given line.

94. Through a given point describe a circle touching two given circles.

95. Describe a circle touching three given circles.

96. In any triangle the intersections of perpendiculars from the angles on the sides, of lines from the angles bisecting the sides, and of perpendiculars bisecting the sides, are in one line, and their distances from one another as 1, 2, 3.

97. Let six points be taken in a plane, A, C, E, in one line, and B, D, F, in another; then shew that the intersections of AB and DE, BC and EF, CD and FA, are in one line.

98. If perpendiculars be dropped from the base-angles of a triangle on the line bisecting the vertical angle, a circle passing through the points of intersection and the foot of the perpendicular from the vertical angle will bisect the base; and the area of the triangle will be equal to the rectangle contained by either perpendicular and the segment of the bisecting line between the angle and the other perpendicular.

99. AC, BE are parallel lines, F, G, H, &c. a series of equidistant points in AC; draw a line through B, cutting EF, EG, EH, &c. in f, g, h, &c., and shew that Bf, Bg, Bh, &c. are in harmonic proportion.

100. If pairs of common tangents be drawn to each two of three given circles, the intersections of each pair will lie in one line.

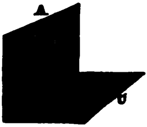
BOOK XI.

DEFINITIONS.

SOLID is that which hath length, breadth, and ss.

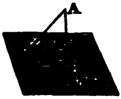
'hat which bounds a solid is a superficies.

A straight line is perpendicular, or at right to a plane, when it makes right angles with traight line meeting it in that plane.

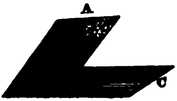


plane is perpendicular to a when the straight lines drawn of the planes, perpendicular to mon section of the two planes, pendicular to the other plane.

he inclination of a straight line to a plane is the ngle contained by that straight d another drawn from the point ch the first line meets the plane point in which a perpendicular plane, drawn from any point of t line above the plane, meets the lane.



The inclination of a plane ne is the acute angle con- by two straight lines drawn y the same point of their n section at right angles to upon one plane, the other upon the other plane.



VII. Two planes are said to have the same, or a like, inclination to one another which two other planes have, when the said angles of inclination are equal to one another.

VIII. Parallel planes are such as do not meet one another though produced.

IX. A solid angle is that which is made by the meeting in one point of more than two plane angles, which are not in the same plane.

X. Equal and similar solid figures are such as are contained by similar planes equal in number and magnitude.

XI. Similar solid figures are such as have all their solid angles equal, each to each, and which are contained by the same number of similar planes.

XII. A pyramid is a solid figure contained by planes that are constituted betwixt one plane and one point above it in which they meet.

XIII. A prism is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to one another, and the others parallelograms.

XIV. A sphere is a solid figure described by the revolution of a semicircle about its diameter, which remains unmoved.

XV. The axis of a sphere is the fixed straight line about which the semicircle revolves.

XVI. The centre of a sphere is the same with that of the semicircle.

XVII. The diameter of a sphere is any straight line which passes through the centre, and is terminated both ways by the superficies of the sphere.

xviii. A cone is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.

If the fixed side be equal to the other side containing the right angle, the cone is called a right-angled cone, if less, an obtuse-angled, and if greater, an acute-angled cone.

xix. The axis of a cone is the fixed straight line about which the triangle revolves.

xx. The base of a cone is the circle described by that side, containing the right angle, which revolves.

xxi. A cylinder is a solid figure, described by the revolution of a right-angled parallelogram about one of its sides, which remains fixed.

xxii. The axis of a cylinder is the fixed straight line about which the parallelogram revolves.

xxiii. The bases of a cylinder are the circles described by the two revolving opposite sides of the parallelogram.

xxiv. Similar cones and cylinders are those which have their axes and the diameters of their bases proportionals.

xxv. A cube is a solid figure contained by six equal squares.

xxvi. A tetrahedron is a solid figure contained by four equal and equilateral triangles.

xxvii. An octahedron is a solid figure contained by eight equal and equilateral triangles.

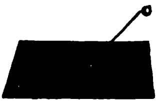
xxviii. A dodecahedron is a solid figure contained by twelve equal pentagons, which are equilateral and equiangular.

XXIX. An icosahedron is a solid figure contained by twenty equal and equilateral triangles.

A. A parallelopiped is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel.

PROP. I. THEOR.

One part of a straight line cannot be in a plane, and another part without it.



If it be possible, let AB, part of the straight line ABC, be in a plane, and the part BC without it: Then, since the straight line AB is in the plane, it can be produced in that plane (1. *Ax.* 2): Let it be produced to D; and let any plane pass through the straight line AD, and be turned about it until it pass through the point C: Now, because the points B, C are in this plane, the straight line BC is in it (1. *Def.* 7): Therefore the two straight lines ABC, ABD are in the same plane and have a common segment AB—which (1. 11. *Cor.*) is impossible: Therefore AB and BC are in the same plane.

Wherefore, *One part of a straight line &c.* Q.E.D.

PROP. II. THEOR.

Two straight lines which cut one another are in one plane; and three straight lines which meet one another are in one plane.

Let the two straight lines AB, CD cut one another in E: AB and CD shall be in one plane; and the three straight lines EC, CB, BE, which meet one another, are in one plane.

Let any plane pass through EB, and be turned about EB, produced if necessary, until it pass through the point C: Then, because the points E, C are in this plane, the straight line EC is in it. In like manner, the straight line BC is in the same plane: And (*Hyp.*) EB is also in it; therefore the three straight lines EC, CB, BE are in one plane: But in the plane in which EC, BE are, in the same are CD, AB (11. 1); therefore AB and CD are in one plane.

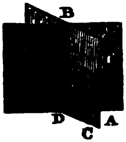
Wherefore, *Two straight lines &c.* Q.E.D.

PROP. III. THEOR.

If two planes cut one another, their common section is a straight line.

Let two planes AB, BC cut one another, and let the line BD be their common section: BD shall be a straight line.

For, if not, from the point B to D, draw, in the plane AB, the straight line BED, and in the plane BC, the straight line BFD: Then the two straight lines BED, BFD have the same extremities, and therefore include a space betwixt them—which (1. *Ax.* 10) is impossible: Therefore BD, the common section of the planes AB, BC, cannot but be a straight line.



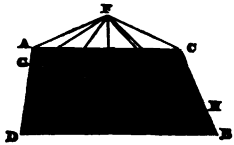
Wherefore, *If two planes &c.* Q.E.D.

PROP. IV. Theor.

If a straight line stand at right angles to each of two straight lines in the point of their intersection, it shall also be at right angles to the plane which passes through them, that is, to the plane in which they are.

Let the straight line EF stand at right angles to each of the two straight lines AB, CD, in E, the point of their intersection: EF shall also be at right angles to the plane passing through AB, CD.

Take AE, EB, CE, ED, all equal to one another; join AD, CB, and through E draw, in the plane in which are AB, CD, any straight line cutting AD, CB in G and H; and from any point F, in EF, draw FA, FG, FD, FC, FH, FB: Then, because the two AE, ED are equal to the two BE, EC, each to each, and that they contain equal angles AED, BEC—therefore the base AD is equal to the base BC, and the angle DAE to the angle EBC: And the angle AEG is equal to the angle BEH; therefore the triangles AEG, BEH have two angles of the one equal to two angles of the other, each to each, and the sides AE, EB, adjacent to the equal angles, equal to one another—therefore GE is equal to EH, and AG to BH: And because AE is equal to EB, and EF common and at right angles to them—therefore the base AF is equal to the base BF; and, in like manner, CF is equal to DF: And because the two sides DA, AF are equal to the two CB, BF, each to each, and the base DF to the base CF—therefore the angle DAF is equal to the



angle CBF: Again, because the two GA, AF are equal to the two HB, BF, each to each, and that they contain equal angles—therefore the base GF is equal to the base HF: And, lastly, because the two GE, EF are equal to the two HE, EF, each to each, and the base GF to the base HF—therefore the angle GEF is equal to the angle HEF, and consequently each of these angles is a right angle: Therefore EF makes right angles with GH, that is, with *any* straight line drawn through E, in the plane passing through AB, CD: Therefore (11. *Def.* 3) EF is at right angles to the plane in which are AB, CD.

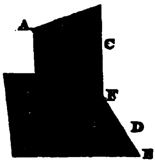
Wherefore, *If a straight line &c.* Q. E. D.

PROP. V. THEOR.

If three straight lines meet all in one point, and a straight line stand at right angles to each of them in that point, the three straight lines shall be in one and the same plane.

Let the straight line AB stand at right angles to each of the straight lines BC, BD, BE, in B the point where they meet: BC, BD, BE shall be in one and the same plane.

For, if not, let, if possible, BD and BE be in one plane, and BC without it; and let a plane pass through AB and BC, the common section of which, with the plane in which are BD and BE, will be a straight line—BF suppose: Then the three straight lines AB, BC, BF are all in one plane, viz. that which passes through AB and BC: And because AB stands at right angles to each of the straight



lines BD, BE, it is also at right angles to the plane passing through them (11. 4): Therefore it makes right angles with every straight line which meets it in that plane, and therefore the angle ABF is a right angle: But the angle ABC (*Hyp.*) is also a right angle; therefore the angle ABF is equal to the angle ABC, and they are both in the same plane—which is impossible: Therefore the straight line BC is not without the plane in which are BD and BE, that is, the three straight lines BC, BD, BE are in one and the same plane.

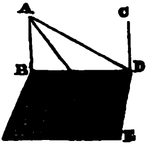
Wherefore, *If three straight lines, &c.* Q. E. D.

PROP. VI. THEOR.

If two straight lines be at right angles to the same plane, they shall be parallel to one another.

Let the straight lines AB, CD be at right angles to the same plane: AB shall be parallel to CD.

Let them meet the plane in the points B, D, and join BD, to which draw DE at right angles, in the same plane; make DE equal to AB, and join BE, AE, AD: Then, because AB is perpendicular to the plane, it makes right angles with every straight line which meets it in that plane: But BD, BE, which are in that plane, do each of them meet AB; therefore each of the angles ABD, ABE is a right angle: In like manner, each of the angles CDB, CDE is a right angle: And because the two AB, BD are equal to the two ED, DB, each to each, and that they contain right angles—therefore the base AD is equal to the base EB: Again, because the two AB, BE are equal to the two



ED, DA, each to each, and the base AE is common to the two triangles ABE, EDA—therefore the angle ABE is equal to the angle EDA: But ABE is a right angle; therefore also EDA is a right angle, and ED perpendicular to DA: But ED is also perpendicular to each of the two BD, DC; therefore ED is at right angles to each of the three straight lines BD, DA, DC, in the point in which they meet; therefore (**xi.** 5) these three straight lines are all in the same plane: But AB is in the plane in which are BD, DA, because any three straight lines which meet one another, are in one plane (**xi.** 2); therefore AB, BD, DC are in one plane: And each of the angles ABD, BDC is a right angle; therefore AB is parallel to CD.

Wherefore, *If two straight lines &c.* Q.E.D.

PROP. VII. THEOR.

If two straight lines be parallel, any straight line, drawn from a point in the one to a point in the other, is in the same plane with the parallels.

Let AB, CD be parallel straight lines, and take any point E in the one and any point F in the other: the straight line which joins E and F shall be in the same plane with AB, CD.

If not, let it be, if possible, without the plane, as EGF; and in the plane ABCD, in which the parallels are, draw the straight line EHF from E to F: Then, since EGF is also a straight line, the two straight lines EGF, EHF include a space between them—which is impossible: Therefore the straight line joining the points E, F is not without the

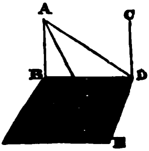
plane in which are the parallels AB, CD, that is, it is in that plane.

Wherefore, *If two straight lines &c.* Q. E. D.

PROP. VIII. THEOR.

If two straight lines be parallel, and one of them be at right angles to a plane, the other also shall be at right angles to the same plane.

Let AB, CD be two parallel straight lines, and let one of them AB be at right angles to a plane: the other CD shall be at right angles to the same plane.



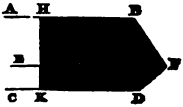
Let AB, CD meet the plane in the points B, D, and join BD; therefore (11. 7) AB, BD, CD are in one plane: in the plane to which AB is at right angles, draw DE at right angles to BD; make DE equal to AB, and join BE, AE, AD: Then, because AB is perpendicular to the plane, it is perpendicular to every straight line which meets it in that plane; therefore each of the angles ABD, ABE is a right angle: And because the straight line BD meets the parallel straight lines AB, CD, the angles ABD, CDB are together equal to two right angles: But of these ABD is a right angle, therefore also CDB is a right angle, and CD is perpendicular to DB: Now, because the two AB, BD are equal to the two ED, DB, each to each, and each of the angles ABD, EDB is a right angle—therefore the base AD is equal to the base EB: Again, because the two AB, BE are equal to the two ED, DA, each to each, and the base AE is common to the two triangles ABE, EDA—therefore the angle ABE is equal to the angle EDA: But ABE is a right angle; therefore also EDA is a right angle, and ED is perpendicular to DA: But ED is also perpendicular to BD; therefore (11. 4) ED is perpendicular to the plane which passes through BD, DA, and makes right angles with every straight line which meets it in that plane: But CD is in the plane passing through BD, DA, because all three are in the plane in which are the parallels AB, CD; therefore ED is at right angles to CD, and therefore also CD is at right angles to DE: But CD is also at right angles to DB; therefore CD is at right angles to the two straight lines DB, DE, in the point of their intersection D, and is therefore at right angles to the plane passing through DB, DE, that is, to the plane to which AB is at right angles.

Wherefore, *If two straight lines &c.* Q.E.D.

PROP. IX. THEOR.

Two straight lines, which are each of them parallel to the same straight line, and not in the same plane with it, are parallel to one another.

Let AB, CD be each of them parallel to EF, and not in the same plane with it: AB shall be parallel to CD.



In EF take any point G, from which draw, in the plane passing through EF and AB, the straight line GH at right angles to EF; and in the plane passing through EF and CD, the straight line GK at right angles to the same EF: Then, because EF is at right angles both to GH and GK, EF is at right angles to the plane GHK which passes through them (11. 4): And EF is parallel to AB; therefore also (11. 8) AB is at right angles to the plane GHK: In like manner, CD is at right angles to the plane GHK: Therefore AB and CD are each them at right angles to the plane GHK, and there (11. 6) they are parallel to one another, that is, AB parallel to CD.

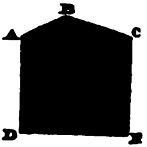
Wherefore, *Two straight lines &c.* Q. E. D.

PROP. X. THEOR.

If two straight lines meeting one another be parallel to tw others that meet one another, and are not in the sa plane with the first two, the first two and the other tw shall contain equal angles.

Let the two straight lines AB, BC, which meet o another, be parallel to the two straight lines DE, EF, that meet one another, and are not in the same plan with AB, BC: the angle ABC shall be equal to the angle DEF.

Take BA, BC, ED, EF all equal to one another, and join AD, BE, CF, AC, DF: Then, because AB is both equal and parallel to DE, therefore AD is both equal and parallel to BE: In like manner, CF is equal and parallel to BE: Therefore AD and CF are each of them equal and parallel to BE, and therefore (11. 9) AD is parallel to CF: And AD is also equal to CF, being each of them equal to BE; therefore also AC is both equal and parallel to DF: And because AB, BC are equal to DE, EF, each to each, and the base AC to the base DF—therefore the angle ABC is equal to the angle DEF.



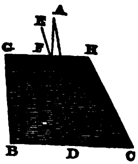
Wherefore, *If two straight lines &c.* Q. E. D.

PROP. XI. Prob.

To draw a straight line perpendicular to a given plane, from a given point without it.

Let A be the given point without the given plane BH: it is required to draw from the point A a straight line perpendicular to the plane BH.

Draw any straight line BC in the plane BH, and from the point A draw AD perpendicular to BC: Then, if AD be also perpendicular to the plane BH, the thing required is done: But, if not, from D draw, in the plane BH, the straight line DE at right angles to BC; and from A draw AF perpendicular to DE, and through F draw GH parallel to BC: Then, because BC is at right angles to ED and DA, BC is at right angles to the plane passing through them (11. 4): And GH is parallel to BC; therefore also (11. 8) GH is at right angles to the plane through ED, DA, and makes right angles with every straight line which meets it in that plane: Therefore GH makes right angles with AF, and AF with GH: But AF also makes right angles with DE; therefore AF makes right angles with each of the straight lines GH, DE, in the point of their intersection, F: Therefore also (11. 4) it is at right angles to the plane through GH, DE, that is, to the plane BH: Wherefore, from the given point A, without the plane BH, the straight line AF has been drawn perpendicular to the plane. Q.E.F.

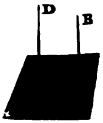


PROP. XII. PROB.

To erect a straight line at right angles to a given plane, from a point given in the plane.

Let A be the given point in the given plane: it is required to erect a straight line from the point A, at right angles to the plane.

From any point B without the plane, draw BC perpendicular to it (11. 11), and from A draw AD parallel to BC: Then, because AD, BC are two parallel straight lines, and one of them BC is at right angles to the given plane, the other AD is also at right angles to it (11. 8): Wherefore a straight line has been erected at right angles to a given plane, from a given point in the same. Q.E.F.



PROP. XIII. THEOR.

From the same point in a given plane, there cannot be drawn two straight lines at right angles to the plane, upon the same side of it: and there can be but one perpendicular to a plane from a point above the plane.

For, if possible, let the two straight lines AB, AC be drawn at right angles to a given plane, from the same point A in the plane, and upon the same side of it: Let a plane pass through BA, AC, and let the common section of this with the given plane be the straight line DAE: Then the straight lines AB, AC, DAE are in one plane: And, because CA is at right angles to the given plane, it makes right angles with the line DAE, which meets it in that plane, and therefore CAE is a right angle:

e manner, BAE is a right angle; therefore the CAE is equal to the angle BAE, and they are in me plane—which is impossible.

›, from a point above a plane, there can be drawn ıe perpendicular to the plane; for, if there could ›, they would be parallel to one another (**xi.** 6)— is absurd.

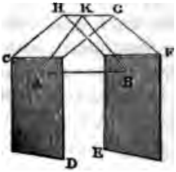
erefore, *From the same point &c.* Q.E.D.

PROP. XIV. THEOR.

, to which the same straight line is perpendicular, are parallel to one another.

the straight line AB be perpendicular to each of lanes CD, EF: these shall be parallel to one ɹr.

, if not, they shall meet one another when pro- : Let their common section straight line HG, in which ɯy point K, and join AK, Then, because AB is per- ɹular to the plane EF, it is ɪdicular to the straight line hich meets it in that plane, erefore ABK is a right angle: manner, BAK is a right angle: Therefore the ıgles ABK, BAK, of the triangle ABK, are equal › right angles—which is impossible: Therefore ınes CD, EF, though produced, do not meet one ɹr, that is, they are parallel.

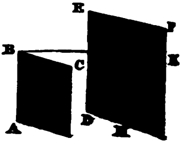


erefore, *Planes &c.* Q.E.D.

PROP. XV. THEOR.

If two straight lines, meeting one another, be parallel to two straight lines which meet one another, but are not in the same plane with the first two, the plane which passes through these is parallel to the plane passing through the others.

Let AB, BC, two straight lines meeting one another, be parallel to DE, EF, which meet one another, but are not in the same plane with AB, BC: the plane through AB, BC, shall be parallel to the plane through DE, EF.



From the point B, draw BG at right angles to the plane through DE, EF, and let it meet it in G; and through G, draw GH parallel to ED, and GK parallel to EF: Then, because BG is perpendicular to the plane through DE, EF, it makes right angles with the lines GH, GK which meet it in that plane, and therefore each of the angles BGH, BGK is a right angle: And because (11. 9) AB is parallel to GH (being each of them parallel to DE), the angles ABG, BGH are together equal to two right angles; and BGH is a right angle; therefore also ABG is a right angle, and GB perpendicular to BA: In like manner, GB is perpendicular to BC: And, because GB stands at right angles to each of the two BA, BC in their point of intersection, B, therefore GB is perpendicular to the plane through BA, BC: And it is also perpendicular to the plane through DE, EF; therefore BG is perpendicular to each of the planes through AB, BC, and DE, EF: But planes, to which the same straight line is perpendicular, are parallel to one another

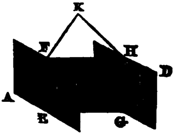
(11. 14); therefore the plane through AB, BC is parallel to the plane through DE, EF.

Wherefore, *If two straight lines &c.* Q. E. D.

PROP. XVI. THEOR.

If two parallel planes be cut by another plane, their common sections with it are parallels.

Let the parallel planes AB, CD be cut by the plane EFHG, and let their common sections with it be EF, GH: EF shall be parallel to GH.



For, if not, EF, GH shall meet, if produced, either on the side of F, H, or E, G: First, let them be produced on the side of F, H, and meet in the point K: Then, since EFK is in the plane AB, every point in EFK is in that plane: and K is a point in EFK; therefore K is in the plane AB: In like manner, K is in the plane CD: Therefore the planes AB, CD, produced, meet one another: But they do not meet, since (*Hyp.*) they are parallel; therefore the straight lines EF, GH do not meet when produced on the side of F, H: Similarly, it may be proved, that EF, GH do not meet when produced on the side of E, G: But straight lines which are in the same plane, and do not meet, though produced either way, are parallel; therefore EF is parallel to GH.

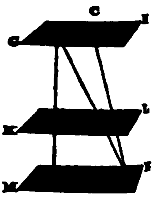
Wherefore, *If two parallel planes &c.* Q. E. D.

PROP. XVII. THEOR.

If two straight lines be cut by parallel planes, they shall be cut in the same ratio.

Let the straight lines AB, CD be cut by the parallel planes GH, KL, MN, in the points A, E, B, C, F, D: AE shall be to EB as CF to FD.

Join AC, BD, AD, and let AD meet the plane KL in X, and join EX, XF: Then, because the two parallel planes KL, MN are cut by the plane EBDX, the common sections EX, BD are parallel (11. 16): In like manner, because the two parallel planes GH, KL are cut by the plane AXFC, the common sections AC, XF are parallel: Now because EX is parallel to BD, a side of the triangle ABD, therefore AE is to EB as AX to XD: And, because XF is parallel to AC, a side of the triangle ADC, therefore AX is to XD as CF to FD: But it was proved that AX is to XD as AE to EB; therefore (5. 11) AE is to EB as CF to FD.



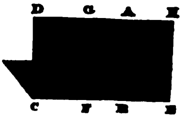
Wherefore, *If two straight lines &c.* Q. E. D.

PROP. XVIII. THEOR.

If a straight line be at right angles to a plane, every plane which passes through it shall be at right angles to that plane.

Let the straight line AB be at right angles to the plane CK: then every plane which passes through AB shall be at right angles to the plane CK.

Let any plane DE pass through AB, and let CE be the common section of the planes DE, CK; take any point F in CE, from which draw FG, in the plane DE, at right angles to CE: Then, because AB



is at right angles to the plane CK, therefore it makes right angles with every straight line which meets it in that plane; therefore it makes right angles with CE, that is, ABF is a right angle: But GFB is also a right angle; therefore FG is parallel to AB: And AB is at right angles to the plane CK; therefore (11. 8) FG is also at right angles to the same plane.

But (11. *Def.* 4) one plane is at right angles to another plane, when any straight line drawn in one of the planes, at right angles to their common section, is also at right angles to the other plane; and it has been shewn that any straight line FG drawn in the plane DE, at right angles to CE, the common section of the two planes, is also at right angles to the plane CK—therefore the plane DE is at right angles to the plane CK.

And, in like manner, it may be shewn that any other plane passing through the straight line AB is at right angles to the plane CK.

Wherefore, *If a straight line &c.* Q. E. D.

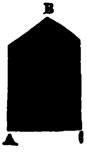
PROP. XIX. THEOR.

If two planes cutting one another be each of them perpendicular to a third plane, their common section shall be perpendicular to the same plane.

Let the two planes AB, BC be each of them perpendicular to a third plane, and let BD be the common section of the two planes AB, BC: BD shall be perpendicular to the third plane.

For, if not, from the point D draw, in the plane AB, the straight line DE at right angles to AD, the common section of the plane AB with the third plane; and from the same point draw, in the plane BC, the straight line

DF at right angles to CD, the common section of t plane BC with the third plane: Then, because t plane AB is perpendicular to the third plane, and D is drawn in the plane AB at right angles to AD their common section, DE is perpendicular to the third plane (II. *Def.* 4): And, in like manner, it may be proved that DF is perpendicular to the third plane: Therefore, from the point D, two straight lines stand at right angles to the third plane, upon the same side of it—which (II. 13) i impossible: Therefore, from the point D, there ca be any straight line at right angles to the third pl except BD the common section of the planes AB, B Therefore BD is perpendicular to the third plane.



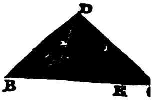
Wherefore, *If two planes &c.* Q. E. D.

PROP. XX. THEOR.

If a solid angle be contained by three plane angles, any t of them are together greater than the third.

Let the solid angle at A be contained by the th plane angles BAC, CAD, DAB: any two of these sh be greater than the third.

If the angles BAC, CAD, DAB be all equal, it evident that any two of them are greater than the thi But, if not, let BAC be that angle which is not le than either of the other two, and is greater than one of them BAD; and at the point A, in the straight line BA, make, in the plane which passes through BA, AC, the angle BAE equal to the angle BAD; and make AE equal to A and through E draw BEC cutting AB, AC, in th points B, C, and join DB, DC.



Then, because AD is equal to AE, and AB is common, the two AB, AD, are equal to the two AB, AE, each to each, and the angle BAD is equal to the angle BAE—therefore the base BD is equal to the base BE: And because BD, DC are together greater than BC, and one of them BD has been proved equal to BE a part of BC, therefore the other DC is greater than the remaining part EC: Then, because AD is equal to AE, and AC common, but the base DC greater than the base EC, therefore (I. 25) the angle DAC is greater than the angle EAC: And, by the construction, the angle BAD is equal to the angle BAE; therefore the angles BAD, DAC are together greater than the angles BAE, EAC, that is, than the angle BAC: But BAC is not less than either of the angles BAD, DAC; therefore BAC, with either of them, is greater than the other.

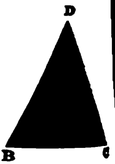
Wherefore, *If a solid angle &c.* Q.E.D.

PROP. XXI. THEOR.

Every solid angle is contained by plane angles, which together are less than four right angles.

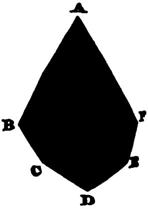
First, let the solid angle at A be contained by three plane angles BAC, CAD, DAB: these shall be together less than four right angles.

In the straight lines AB, AC, AD, take any points B, C, D, and join BC, CD, DB: Then, because the solid angle at B is contained by the three plane angles CBA, ABD, DBC, any two of these are together greater than the third; therefore the angles CBA, ABD are together greater than the angle CBD: For the same reason, the angles BDA, ADC are together greater than the angle BDC, and the angles DCA, ACB are together greater than the angle DCB : Therefore th six angles CBA, ABD, BDA, ADC, DCA, ACB a together greater than the three angles CBD, BDC, DCB : But the three angles CBD, BDC, DCB a together equal to two right angles ; therefore the si angles CBA, ABD, BDA, ADC, DCA, ACB are togethe greater than two right angles : And, because the three angles of each of the triangles ABC, ACD, ADB are together equal to two right angles, therefore the nine angles of these three triangles are together equal to six right angles : And of these, the six angles CBA, ABD, BDA, ADC, DCA, ACB are together greater than two righ angles ; therefore the remaining three angles BAC, BAD, CAD, which contain the solid angle at A, ar together less than four right angles.



Next, let the solid angle at A be contained by any number of plane angles BAC, CAD, DAE, EAF, FAB: these shall be together less than four right angles.

Let the planes in which the angles are be cut by a plane, and let the common sections of it with those planes be BC, CD, DE, EF, FB : Then, because the solid angle at B is contained by three plane angles CBA, ABF, FBC, any two of which are greater than the third, therefore the angles CBA, ABF are together greater than the angle FBC : In like manner, at each of the points C, D, E, F, the two angles, which are at the bases of the triangles having the common vertex A, are together greater than the third angle at the same point, which is one of the angles of the polygon BCDEF : Therefore



all the angles at the bases of the triangles are together greater than all the angles of the polygon: But all the angles of the triangles are together equal to twice as many right angles as there are triangles, that is, as there are sides in the polygon BCDEF; and all the angles of the polygon, together with four right angles, are also equal to twice as many right angles as there are sides in the polygon—therefore all the angles of the triangles are equal to all the angles of the polygon together with four right angles: But it has been shewn that all the angles at the bases of the triangles are together greater than all the angles of the polygon; therefore the remaining angles of the triangles, viz. those at the vertex, which contain the solid angle at A, are together less than four right angles.

Wherefore, *Every solid angle &c.* Q.E.D.

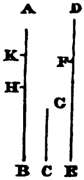
23

BOOK XII.

LEMMA.

If from any given magnitude there be taken a magnitude greater than its half, and from the remainder a magnitude greater than its half, and so on, there shall at length remain a magnitude less than any other given magnitude less than the former.

Let AB and C be the two given magnitudes, of which AB is the greater: if from AB there be taken a magnitude greater than its half, and from the remainder a magnitude greater than its half, and so on, there shall at length remain a magnitude less than C.



Take of C such a multiple DE as shall be greater than AB; and let DE be divided into parts, DF, FG, GE, each equal to C: From AB take BH, greater than its half, and from the remainder AH take HK greater than its half, and so on, till the number of the magnitudes BH, HK, KA is equal to the number of the magnitudes DF, FG, GE: Then, because DE is greater than AB, and that EG taken from DE is not greater than its half, but BH taken from AB is greater than its half, therefore the remainder DG is greater than the remainder AH: And, in like manner, DF is greater than AK: but DF is equal to C; therefore also C is greater than AK, that is, AK is less than C.

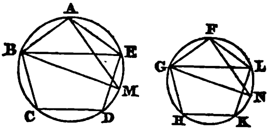
Wherefore, *If from any given magnitude &c.* Q.E.D.

N.B. If only the halves be taken away, the same thing may in the same way be demonstrated.

PROP. I. THEOR.

Similar polygons inscribed in circles are to one another as the squares of their diameters.

Let the similar polygons ABCDE, FGHKL, be inscribed in circles, whose diameters are BM, GN: the polygon ABCDE shall be to the polygon FGHKL as the square of BM to the square of GN.



Join AM, BE, FN, GL: Then, because the polygons are similar, therefore the angle BAE is equal to the angle GFL, and BA is to AE as GF to FL: Therefore the two triangles BAE, GFL, having one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, are equiangular, and therefore the angle AEB is equal to the angle FLG: But the angle AEB is equal to the angle AMB, because they stand upon the same circumference, and, for the same reason, the angle FLG is equal to the angle FNG; therefore the angle AMB is equal to the angle FNG: And the right angle BAM is equal to the right angle GFN; therefore the triangles ABM, FGN are equiangular, and therefore BA is to GF as BM is to GN: Therefore also (5. *Def.* 10, and 5. 22) the duplicate ratio of BA to GF is the same with that of BM to GN: But (6. 20) the polygon ABCDE is to the polygon FGHKL in the duplicate ratio of BA to GF, and the square of BM is to the square of GN in the duplicate ratio of BM to GN; therefore the polygon ABCDE is to the polygon FGHKL as the square of BM to the square of GN.

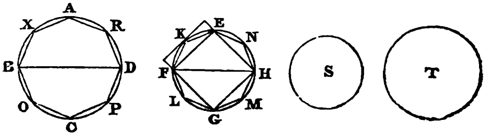
Wherefore, *Similar polygons &c.* Q. E. D.

PROP. II. THEOR.

Circles are to one another as the squares of their diameters.

Let ABCD, EFGH, be two circles, BD, FH, their diameters: the circle ABCD shall be to the circle EFGH as the square of BD to the square of FH.

For, if not, the square of BD must be to the square of FH as the circle ABCD is to some space either less than the circle EFGH, or greater than it: First, if possible, let it be to a space S, which is less than the circle EFGH; and in this circle inscribe the square EFGH: Then this square will be greater than half of the circle EFGH: (For, if through the points E, F, G, H, there be drawn lines touching the circle, it will be manifest, by joining EG, that the square EFGH is half of the square described about the circle, and therefore is greater than half of the circle:) Bisect the circumferences EF, FG, GH, HE in the points K, L, M, N, and join EK, KF, FL, LG, GM, MH, HN, NE: Then each of the triangles EKF, FLG, GMH, HNE, is greater than half of the segment upon the same base: (For, if through the points K, L, M, N, there be drawn lines touching the circle, and the parallelograms upon the lines EF, FG, GH, HE, be completed, each of the triangles will be half of the parallelogram upon the same base, and therefore greater than half of the segment upon the same base:) Therefore the sum of these triangles will be together greater than half the



sum of the segments, that is, than half the remainder of the circle, when the square EFGH is taken from it: Let these triangles also be taken away; then, by bisecting the circumferences EK, KF, &c., and forming triangles, as before, and taking them away, and so on continually, there will at length remain segments of the circle, which (*Lemma*) shall be together less than any given magnitude less than the circle EFGH, and therefore shall be together less than the excess of the circle EFGH above the space S: Let the segments EK, KF, &c. be those which remain, and are together less than the excess of the circle EFGH above the space S; then the rest of the circle, that is, the polygon EKFLGMHN, is greater than the space S.

Now, if in the circle ABCD be described a polygon AXBOCPDR similar to the polygon EKFLGMHN, then (XII. 1) the first polygon is to the second as the square of BD to the square of FH, that is, (*Hyp.*) as the circle ABCD is to the space S: But the first polygon AXBOCPDR is manifestly less than the circle ABCD, in which it is inscribed; therefore the second polygon, EKFLGMHN, is also less than the space S: But it has been proved to be greater than it—which is impossible: Therefore the square of BD is not to the square of FH as the circle ABCD is to any space less than the circle EFGH; and so also, in like manner, the square of FH is not to the square of BD as the circle EFGH to any space less than the circle ABCD.

Next, if possible, let the square of BD be to the square of FH as the circle ABCD to a space T, greater than the circle EFGH: Then, inversely, the square of FH is to the square of BD as the space T to the circle ABCD: But as the space T is to the circle ABCD so is the circle EFGH to some space, which must be less than the circle ABCD, (since (*Hyp.*) the space T is greater than the circle EFGH;) therefore the square of FH is to the square of BD as the circle EFGH is to some space less than the circle ABCD—which has been just shewn to be impossible: Therefore the square of BD is not to the square of FH as the circle ABCD to any space greater than the circle EFGH, nor, as has been shewn before, to any space less than the circle EFGH; therefore the space of BD must be to the square of FH as the circle ABCD to the circle EFGH.

Wherefore, *Circles are to one another as the squares of their diameters.* Q.E.D.

PROBLEMS.

BOOKS V. XI. XII.

AND MISCELLANEOUS.

1. If any three lines be drawn making equal angles with the sides of a triangle toward the same parts, they will form a triangle similar to the former.

2. The vertical angle BAC of a triangle is bisected by a line on which the perpendiculars BD, CE are drawn: bisect the base BC in F, and shew that FD = FE.

3. From the vertex of an isosceles triangle, two lines are drawn to the opposite angles of the square upon the base: shew that the line, which joins their intersections with the diagonals, is parallel to the base.

4. In the figure I. 47, shew that the lines BC, FK, GH, meet in a point.

5. If England be a triangle, whose base is 200 miles and altitude 300 miles, and if the population be 20 millions, how many persons are there to an acre?

6. The sum of the perpendiculars, let fall from any point within an equilateral triangle upon the sides, will be equal to that let fall from one of its angles upon the opposite side. Is this true when the point is in one side of the triangle? How must the statement be made when the point is without the triangle?

7. If P be any point in the plane of a parallelogram, ABCD, shew that the triangle PBD is equal to the sum of the triangles PAB, PBC.

8. Let a, b, c represent the sides opposite to the angles A, B, C, of any triangle; then if A be a right angle,

and h the length of the perpendicular from A on BC, the triangle whose sides are $b+c$, h, $a+h$, is also right-angled.

9. Express AD, the perpendicular upon the base BC of *any* triangle, in terms of a, b, c; and shew that if S represent the semisum of the sides, the area $=\sqrt{\{S(S-a)(S-b)(S-c)\}}$.

10. If an equilateral triangle have its angular points in three parallel lines, of which the middle one is distant from the outside ones by b, c, its side will be

$$2\sqrt{\frac{b^2+bc+c^2}{3}}.$$

11. If the sides of a pentagon be produced both ways to meet, shew that the sum of the angles at the points of intersection is equal to two right angles; and, generally, if a polygon have $n+4$ sides, shew that the sum of the angles thus formed is equal to $2n$ right angles.

12. Shew that the triangle, square, and hexagon, are the only regular polygons, by which space can be completely filled up about a point.

13. Shew how the squares in (I. 47) may be dissected, so that its truth may be made to appear by superposition of the parts.

14. If AD be drawn to any point D in the base of the triangle ABC, shew that

$$AB^2.CD+AC^2.BD=AD^2.BC+BC.BD.CD.$$

15. Find the locus of a point, when (i) the sum, and (ii) the difference, of its squares from two given points is constant.

16. C is any point in AB; find a point D (i) in CB, (ii) in CB produced, such that CD may be a mean proportional between AD, BD.

17. AB is divided in extreme and mean ratio at C: in BC (the greater segment) produced take BD=BC, and in BC take BE=AC; then will AD, AE be also similarly divided in B, C.

18. ABCD is a trapezium, AD parallel to BC: shew that $AC^2 \sim BD^2 : AB^2 \sim CD^2 :: BC + AD : BC \sim AD$.

19. Divide algebraically a given line a into two parts, such that the rectangle of the whole and one part may be equal to the square of the other part. Deduce Euclid's construction from one solution, and explain the other.

20. Two circles, whose radii are as 2 : 3, touch each other internally, and through the centre of the smaller circle a line is drawn perpendicular to the common diameter: shew that tangents drawn to this circle, from the points where the line cuts the larger circle, are perpendicular to each other.

21. In any inscribed polygon of an even number of sides, the sum of the 1st, 3rd, 5th, &c. angles is equal to the sum of the 2nd, 4th, 6th, &c. angles.

22. BD, CD, are drawn perpendicular to the sides AB, AC of a triangle, and CFE, perpendicular to AD, cuts AB in E: shew that the triangles ABC, ACE are similar.

23. AB, the line of centres of two circles, whose radii are R, r, is divided in C, so that

$$AB : R+r :: R-r : AC-BC :$$

shew that the tangents drawn to the circle from any point in CD, perpendicular to AB, are equal.

24. Describe a rectangle equal to a given square, and having the difference of its adjacent sides equal to a given line.

25. If a circle be described about a triangle ABC, and perpendiculars from the angles on the sides cut the circle in D, E, F, the arcs DE, DF, EF, will be bisected in C, B, A.

26. If any two circles touch in O, and lines, POP′, QOQ′, be drawn at right angles to each other, then, A, A′, being the points in which the line of centres cuts the circles, $PP'^2 + QQ'^2 = AA'^2$.

27. AOB is a quadrant: draw any chord $QRrq$ parallel to AB, cutting OA, OB in R, r, and shew that $QR^2 + Qr^2 = AB^2$.

28. Find a point such that tangents drawn from it to touch two given circles shall contain a given angle.

29. If from any point in the circumference of a circle perpendiculars be drawn upon the sides of the inscribed triangle, the three points of intersection will be in the same line.

30. AB, AC, DE, DF, are four lines, which form by their intersections, three and three together, four triangles: shew that the circles which circumscribe them will pass through one and the same point.

31. If the base of a triangle be produced both ways, so that each part produced may equal the adjacent side, and through the extremities of the parts produced and vertex a circle be described, the line joining the vertex and centre of the circle will bisect the vertical angle.

32. No triangle can be inscribed in a square greater than half the square.

33. If R, r be the radii of the circumscribed and inscribed circles of a triangle, shew that its area

$$= \frac{abc}{4R}, \text{ or } = \frac{(a+b+c)\,r}{2},$$

a, b, c representing the sides.

34. The sides of a triangle are in Ar. Prog, a, a' being the longest and shortest sides: if R, r be the radii of the circumscribed and inscribed circles, shew that $6Rr = aa'$.

35. ABCD is an inscribed quadrilateral: shew that AC : BD :: BA.AD + BC.CD : AB.BC + AD.DC.

36. If the circumference of a circle be divided into five equal parts in the points A, B, C, D, E, and these be joined two and two, that is, AC, CE, EB, BD, DA, shew that the angles of the stellated figure thus formed are together equal to two right angles.

37. If the number of parts in [36] be m, and the points of division be joined n and n together, n being less than $\frac{1}{2}m$, shew that the $\angle$s of the stellated figure thus formed are together equal to $2\,(a-2b)$ right angles, where a, b are the prime factors of m and n. How may this result be applied to the case when n is greater than $\frac{1}{2}m$?

38. Construct the figures for stellated pentagons, hexagons, heptagons, and octagons; and determine how many forms are possible in each case.

39. If AB be divided in extreme and mean ratio at C, shew that the ratio of AC : CB is incommensurable, but nearly $=1\frac{5}{8}$.

40. A ratio of greater inequality is diminished, and of less inequality increased, by adding the same quantity to both its terms.

41. Apply a property of the circle (III. 15) to shew that, if four lines are proportionals, the sum of the greatest and least will be greater than the sum of the other two.

42. If, from any point O within a triangle, there be drawn Oa, Ob, Oc, to the sides, and from the angles Aa', Bb', Cc', be drawn parallel to these, shew that

$$\frac{\mathrm{O}a}{\mathrm{A}a'}+\frac{\mathrm{O}b}{\mathrm{B}b'}+\frac{\mathrm{O}c}{\mathrm{C}c'}=1.$$

43. Construct a right-angled triangle, having given the sum of the base and perpendicular, and the sum of the base and hypothenuse.

44. If A be the area of any triangle, shew that the area of a triangle, whose angular points divide the sides of the former in the ratio of $n:1$, will be $\frac{n^2-n+1}{(n+1)^2}.\mathrm{A}$.

45. From the right angle A of a triangle a perpendicular Ap is dropped upon BC, pq on AB, qr on BC, &c.: shew that A$p+pq+qr+$&c. : AB :: AB + BC : AC.

46. CD is a chord of any circle, parallel to any given line AB; AC cuts the circle again in E, and BE in F: shew that DF cuts the line AB in a fixed point G.

47. In the sides AB, AC of a triangle, take two points M, N, and, in MN, a point P, such that

$$\frac{BM}{AM} = \frac{AN}{CN} = \frac{PM}{PN};$$

and prove that the triangle BPC is double of the triangle AMN.

48. ABC is an inscribed triangle, AD, AE are lines drawn to BC, parallel to the tangents at C, B: shew that BE : CD :: $AB^2 : AC^2$.

49. DEF is a circle inscribed in the triangle ABC, touching the sides BC, AC, AB, in D, E, F; HIK a circle touching AB in K, and CB, CA produced in H, I: in CH take CL = CA, and in CI take CM = CB, and shew that FK = AM.

50. B, C are any two points in AD; find a point such that lines drawn from it to A, B, C, D, shall contain equal angles.

51. One side of a polygon is divided into n parts, on each of which is described a figure similar and similarly situated to the given figure: shew that the sum of the peripheries of the smaller figures is equal to that of the larger, and, if the parts be equal, that the areas of the smaller figures are together equal to $\frac{1}{n}$th of the area of the larger.

52. The regular inscribed hexagon is a mean proportional between the inscribed and circumscribed equilateral triangles of any circle.

53. If on the diameter of a semicircle two equal semicircles be described, and in the space between the three circumferences a circle be inscribed, its diameter will be to that of the equal circles :: 2 : 3.

54. If from the angles C, D of a quadrilateral, ABCD, and from E, the intersection of its diagonals, perpendiculars CF, DG, EH, be dropped upon AB, its area

$$=\tfrac{1}{2}\,\text{AB}.\,\frac{\text{CF.DG}}{\text{EH}}\,.$$

55. If A*a*, B*b*, C*c* be drawn, bisecting the angles of a triangle, shew that

$$\text{A}b.\text{B}c.\text{C}a=\frac{a^2b^2c^2}{(a+b)\;(b+c)\;(c+a)}\,.$$

56. Given *a*, *b*, *c* the lengths of the chords of three arcs, which together make up a semicircle, shew that its diameter may be found from the equation

$$x^3-(a^2+b^2+c^2)\,x=2abc.$$

57. If P, *p* be the areas of regular polygons of *n* sides, circumscribing, and inscribed in, a given circle, and *Q*, *q* the areas of corresponding polygons of 2*n* sides, shew that $P:q::q:p$, and $P:Q::p+q:2p$.

58. Draw a perpendicular to two given lines, not in the same plane.

59. Two planes being perpendicular to each other, draw a third, perpendicular to both.

60. From two given points, draw equal lines to the same point of a given line, not in the same plane with them.

61. If three lines, not in the same plane, be equal and parallel, shew that the triangles formed by joining their adjacent extremities are equal and their planes parallel.

62. If two lines are parallel, the common section of any two planes passing through them will be parallel to either.

63. Having three points given in a plane, find a point above the plane equidistant from either of them.

64. Describe a circle which shall touch two given planes and pass through a given point.

65. If a line be perpendicular to a plane, its projection on any other plane will be perpendicular to the line of intersection of the planes.

66. Two points are taken on two walls, which meet at an angle: find the shortest line which joins them.

67. B, D are points at equal distances from the ends of the arc of a quadrant AOC, and BG, DH are perpendiculars on OC: shew that the figure BGHD is equal to the sector OBD.

68. If semicircles be described on any two segments of the diameter of a circle, the area between the three circumferences will be equal to the area of a circle, whose diameter is the mean proportional between the segments.

69. Similar arcs of different circles are as the radii, and similar sectors as the squares of the radii.*

70. Shew that the angles at the centre (or circumference) in different circles are proportional directly to the arcs subtending them and inversely to the radii; and hence that if a circle roll within another of twice its size, any point in its circumference will trace out a diameter of the first.

71. Sectors of different circles are equal, when their angles are inversely as the squares of their radii.

72. Similar segments of circles are proportional to the squares of the lines on which they stand. Hence shew that, if ABC be an isosceles triangle, right-angled at A, BDEC a semicircle on BC, and BFC a circle described with centre A, and radius AB, the segment BCF is equal to the segments BAD, ACE.

73. Any sector of a circle is equal to a triangle, whose height is the radius, and base the subtending arc.

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* It may be assumed that in any circle, semicircumference : radius :: area : square of radius; hence since the areas of different circles are proportional to the squares of the radii (XII. 2), their circumferences are proportional to the radii.

74. If AB be a circular arc, centre O, and AD be drawn perpendicular to BO, and the arc AC be taken equal to AD, then the sector BOC is equal to the segment ACB.

75. The difference of two similar sectors AOB, A′OB′, equals the area of a rectangle, whose sides are AA′, the difference of the radii, and *ab*, a concentric and similar arc midway between AB and A′B′.

76. If semicircles be described on the three sides of a right-angled triangle towards the interior, the difference between the sum of the circular segments thus standing upon the exterior of the sides and the sum of those upon the segments of the base, equals the space intercepted by the circumferences on the sides.

77. If a line be placed in a circle, and on the radius through one extremity a circle be described, the segments of the two circles cut off by the line will be in the ratio of 4 : 1.

78. Semicircles OEDA, OFDB, are described on the radii of a quadrant, OACB: shew that (i) A, B, D are in a line; (ii) the areas ACBD, OEDF are equal; (iii) the areas OFDA, OEDB, are each one-fourth of the square of the radius.

79. AB, CD are diameters of a circle, intersecting at right angles in O: with radius DA or DB describe an arc AEB, and shew that the areas of the lune AEBC and of the triangle DAB are equal.

80. AD is perpendicular on the hypothenuse of a right-angled triangle ABC: shew that the circles inscribed in the triangles ABD, ACD, will be proportional to those triangles.

81. A series of circles are described, touching each other successively and each of two given lines: if OA be the distance of the centre of the outermost circle from the intersection of the lines and OB its radius, shew that the sum of all the circles : outermost circle :: $(OA+OB)^2 : 4OA . OB$.

82. If the diameter AB of a circle be divided into n equal parts, in the points P_1, P_2, &c., and upon AP_1, AP_2, &c. semicircles be described on one side of the diameter, and also upon BP_1, BP_2, &c. on the other side, the perimeter of any one of the figures $AP_{m-1}BP_m$ is equal to the circumference of the circle, and its area $\frac{1}{n}$-th of the area of the circle.

83. Upon the three sides of a triangle right-angled at A, describe semicircles towards the same parts, AEB, ADC, BDC; and shew that the difference of the figures ADBE, CD, is equal to the triangle ABC.

84. The sides of a square are bisected, and by joining the bisections another is inscribed, another in this, and so on: shew that all the inscribed squares are together equal to the given one.

85. Shew that all lines drawn from an external point to touch a given sphere are equal; and thence that, if a tetrahedron can have a sphere inscribed touching its six edges, the sum of every two opposite edges is the same.

86. There can be only five regular solids.

87. If A*a*, B*b*, C*c*, be lines drawn from the angles of a triangle to the opposite sides, (i) bisecting the sides, (ii) perpendicular to the sides, (iii) bisecting the angles, (iv) to the points of contact of inscribed circle, then, in each case, $Ab \, . \, Bc \, . \, Ca = Ac \, . \, Ba \, . \, Cb$.

88. If A*a*, B*b*, C*c*, be lines drawn from the angles of a triangle to the sides through any point O, then $Ab \, . \, Bc \, . \, Ca = Ac \, . \, Ba \, . \, Cb$; and, conversely, if this equality obtain, the lines A*a*, B*b*, C*c*, will pass through a point.

89. If two sides AB, AC of a triangle are divided proportionally in *c*, *b*, and the third, BC, bisected in *a*, the lines A*a*, B*b*, C*c*, will meet in a point.

90. In 6. 2, fig. 1, if BE, CD meet in F, shew that

AF produced will bisect BC in G, and, if $AB = n \, . \, AD$, find the ratio of AF to AG.

91. If any line be drawn cutting the sides BC, AC, AB of a triangle in a, b, c, shew that $Ab.Bc.Ca = Ac.Ba.Cb$; and, conversely, if this equality obtain, the points a, b, c lie in one line.

92. If one side of a triangle, AB, be produced to D, and the other, AC, shortened by CF, equal to BD, the line DF will divide BC in the ratio of AC to AB, and BC will divide DF in the ratio of AF to AD.

93. Lines are drawn from the angles of a triangle ABC, through any point, to meet the sides in a, b, c, and through these points a circle is described, cutting the same sides again in a', b', c': shew that Aa', Bb', Cc' meet in a point.

94. If through a given point within a triangle lines be drawn from the angles to the opposite sides, and the points of section be joined, the three first drawn lines will be harmonically divided: and, conversely, if any line AH be divided harmonically in G, D, and F be a point in any other line AB, the intersections E, C, of FG, FD, with BD, BH, will be in a line.

95. If X be an object inaccessible to an observer A, shew that, taking in the direction XA any point B, and in any other direction through B any two points C, D, and E the point of intersection of the lines AD, CX,

$$AX = \frac{AB \, . \, CD \, . \, EA}{BC \, . \, DE - AE \, . \, CD} \, .$$

96. Under the same circumstances as in [95] describe any triangle ABC, and through X draw any line, cutting AC, BC in D, E: then, if the line through C and the intersection of AE, BD, cut AB in F, $AX = \dfrac{AB \, . \, AF}{BF - AF}$.

97. The sides AB, AC, BC, of an inscribed triangle are produced to cut a given line in m, n, p; if t_m, t_n, t_p, be the lengths of the tangents drawn from m, n, p, to the circle, shew that $t_m . t_n . t_p = Am . Bp . Cn = An . Bm . Cp$.

98. In every inscribed triangle, the intersection of the sides and tangents at the angles opposite are in a line.

99. In every inscribed hexagon, the intersections of the opposite sides taken two and two are in one line.

100. If a hexagon be described about a circle, the lines joining the opposite angles intersect in a point.

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